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Traffic Flow on Networks

Conservation Laws Models

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To our parents Loris, Maria, Giuliano and Margherita

Preface

This book is devoted to macroscopic models for traffic on a network, with possible applications to car traffic, telecommunications and supply-chains.

The problem of modelling car traffic has a long history dating back to the beginning of the 20th century. Many methods have been developed resorting to different approaches ranging from microscopic ones, taking into account each single car, to kinetic and macroscopic fluid-dynamic ones, dealing with averaged quantities. See for instance [9, 38, 63, 68, 89, 91, 106]. In particular, in the 50s a fluid-dynamic model was introduced by James Lighthill and Gerald Whitham (and independently P. Richards). They thought that the equations describing the flow of water could also describe the flow of car traffic. The obtained model consists of a single conservation law and is on one side simple enough to permit a complete understanding, on the other side reach enough to detect important phenomena as queue formation and evolution.

Nowadays, the exponentially increasing number of circulating cars in modern cities renders the problem of traffic control of paramount importance. The presence of hard congestions on urban networks may have dramatic implications affecting productivity, pollution, life-style etc. Therefore, solutions to these new challenges will be of great socio-economical impact.

Starting from classical (Lighthill-Whitham-Richards) and less classical (Aw-Rascle [12]) fluid-dynamic approaches to describe car traffic on a single road, the book develops an original theory to deal with arbitrarily complex networks, thus including the modelling of road networks of big cities or of highways systems of big states. Also, the book provides possible extensions to treat data flows on telecommunication networks.

The main advantages of the present approach, with respect to existing ones are the following:

- The model can be used to study the evolution of network congestions as consequence of sudden changes or special situations as accidents, demonstrations, floods etc. In fact, the fluid-dynamic models are completely evolutive, thus they are able to describe the traffic situation of a network at every instant of time. This overcomes the difficulties encountered by many static models.
- An accurate description of queues formation and evolution on the network and, as a consequence, an accurate evaluation of travelling times is possible. Indeed, one of the main features of conservation laws is the presence of shock waves in solutions describing queues tails in traffic flow.
- The theory permits the development of efficient numerical schemes also for very large networks. This is possible since traffic at junctions is modelled in a simple and computationally convenient way (resorting to a linear programming problem).
- Real urban networks are well described. For example: traffic lights, traffic circles, complex junctions etc. This is well illustrate in Chapter 8, where a comparison of a traffic light with a traffic circle is developed in detail.
- Tests with real data are convenient and easy to implement. The number of parameters in the model is low, thus overcoming many of the difficulties of microscopic and mesoscopic models.

The book is suitable both for researchers interested in traffic problems and for graduate students in mathematics, physics and engineering. The required background of conservation laws is presented in Chapter 2, while some technicalities are postponed to Appendices.

The book contains exercises (to Chapters 2, 3, 4, 5), bibliographical notes (to Chapters 2, 3, 5, 6, 7) and open problems (to Chapters 4, 6, 7).

The text can be used both for a short half semester course based on simple models for traffic, and for a long one semester course.
A complete course on fluid-dynamic models for urban traffic can be based on all Chapters excluding Chapter 9. In this case, students will benefit from a good view on existing macroscopic models and a robust education on their extension to complex networks, ranging from modelling to analysis and numerics.

For a short course on urban traffic, the teacher may use Chapters from 2 to 5 introducing students to basics. Then one of Chapters 6, 7, 8 or 10, may complete the course focusing on the preferred specific issue. Finally, a short course on telecommunication networks is covered by Chapters 2, 4, 5 and 9.

The authors are greatly indebted to their families for the constant support received during the drafting of this book. We want also to thank Ciro D'Apice and Rosanna Manzo for having written Chapter 9 about traffic in a telecommunication network. We are also greatly indebted with Alberto Bressan, Gabriella Bretti, Yacine Areski Chitour, Giuseppe Maria Coclite, Roberto Natalini and Antonio Sgalambro, which contributed in various ways.

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Readers' contributions via e-mail addresses `mauro.garavello@unimib.it`, `b.piccoli@iac.cnr.it`, are welcome.

Milano, March 2006 — *Mauro Garavello*
Roma, March 2006 — *Benedetto Piccoli*

Notation

We collect here a list of notation commonly used in this book.

$\mathbb{N}$	the set of natural numbers including 0.
$\mathbb{Q}$	the set of rational numbers.
$\mathbb{R}$	the set of real numbers.
L^p_{loc}	the set of functions locally L^p integrable.
C^0	the set of continuous functions.
C^0_c	the set of continuous functions with compact support.
C^1	the set of continuous and differentiable functions.
BV	the set of bounded variation functions.
Tot.Var.	the total variation of a function.

Contents

1

Introduction

The aim of this book is to present a new theory, based on conservation law models, for traffic flow on networks. The thematic is suitable for various applications including: urban car traffic [27], data flows on telecommunication networks [41] and supply chains models [8, 53].

The main inspiration is that of understanding traffic behavior in urban context in order to answer to several questions: where to install traffic lights or stop signs; how long the cycle of traffic lights should be; where to construct entrances, exits, and overpasses. The aims of this analysis are principally represented by the maximization of cars flow, and the minimization of traffic congestions, accidents and pollution.

Classically, the network models of transportation systems are assumed to be static, but these models do not allow a correct simulation of heavily congested urban road networks. For this reason, traffic engineers have been studying dynamic traffic assignment or *within-day* models, thus rendering necessary the use of time advancing mathematical models (traffic simulation models). These models, principally created from static network traffic assignments, can be roughly classified in microscopic, mesoscopic and macroscopic (see [10] and the references therein). The main problems of this approach consist in the fact that it does not properly reproduce the backward propagation of shocks and in the difficulty of collecting experimental data to test the models. Various other ideas have been developed by researchers studying traffic from other perspectives, see for instance [9, 38, 63, 68, 89, 91, 106]. In many cases, the attention was focused on a single road or on small portions of an urban network.

In the 1950s James Lighthill and Gerald Whitham, two experts in fluid-dynamics, (and independently P. Richards) thought that the equations describing the flow of water could also describe the flow of car traffic. These equations in fluid dynamics are a set of partial differential equations known as the Euler or Navier-Stokes equations, expressing the conservation of mass, momentum and energy. The basic idea is to look at large scales so to consider cars as small particles and their density as the main quantity to be considered. In any case, it is reasonable to assume the conservation of the number of cars,

thus leading again to a conservation law. As traffic jams display sharp discontinuities, there is a correspondence between traffic jams and shock waves. Therefore, fluid-dynamic models for traffic flow seem the most appropriate to detect some phenomena as shocks formation and propagation on roads, since solutions can develop discontinuities in a finite time even starting from smooth initial data (see [19]).
This nonlinear framework, based on the conservation of cars, is described by the equation:

$$\partial_t \rho + \partial_x f(\rho) = 0, \tag{1.0.1}$$

where $\rho = \rho(t,x)$ is the density of cars, with $\rho \in [0, \rho_{max}]$, $(t,x) \in \mathbb{R}^2$ and ρ_{max} is the maximum density of cars on the road; $f(\rho)$ is the flux, which can be written $f(\rho) = \rho v$ with v the average velocity of cars. In most cases one assumes that v is a function of ρ only, thus also $f = f(\rho)$ and its graph is called the *fundamental diagram*. We make this assumption, moreover, for simplicity, we let f be concave and have a unique maximum $\sigma \in]0, \rho_{max}[$ (the non concave case is discussed along the book).

A simple choice for the velocity is that of a linear decreasing function:

$$v(\rho) = v_{max}\,(\rho_{max} - \rho), \tag{1.0.2}$$

thus the resulting flux is given by:

$$f(\rho) = v_{max}\,\rho\,(\rho_{max} - \rho),$$

see Figure 1.1.

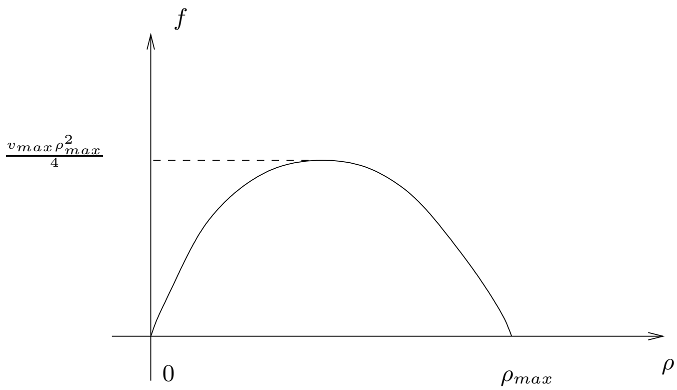


Fig. 1.1. The flux function when the velocity is given by (1.0.2).

To illustrate the behavior of solutions to (1.0.1), we start focusing on the simplest example of junction (or of network) which is a traffic light. Assume that the traffic light is positioned at $x = 0$ and consider the initial density of cars given by:

$$\rho_0(x) = \begin{cases} \rho_{max}, & \text{if } x \le 0, \\ 0, & \text{if } x > 0, \end{cases} \tag{1.0.3}$$

so that the road before the light is full of cars and empty in front of the light. This typical datum corresponds to a situation in which the traffic light is red, so that cars are in a queue at the light. Assume that the green starts at time $t = 0$, then cars start to pass through. This is well detected from the solution to (1.0.1) with initial datum given by (1.0.3). In fact the evolution is given by:

$$\rho(t,x) = \begin{cases} \rho_{max}, & \text{if } x < f'(\rho_{max})t, \\ (f')^{-1}(\frac{x}{t}), & \text{if } f'(\rho_{max})t < x < f'(0)t, \\ 0, & \text{if } x > f'(0)t; \end{cases}$$

see Figure 1.2. For a fixed time $t > 0$, the solution is equal to ρ_{max} to the left of the point $f'(\rho_{max})t$, hence there is a queue beyond this point; the solution vanishes to the right of the point $f'(0)t$, hence no car reached yet this point; finally, in the middle, there is a decreasing density, which is the effect of progressive acceleration of cars at the green light.

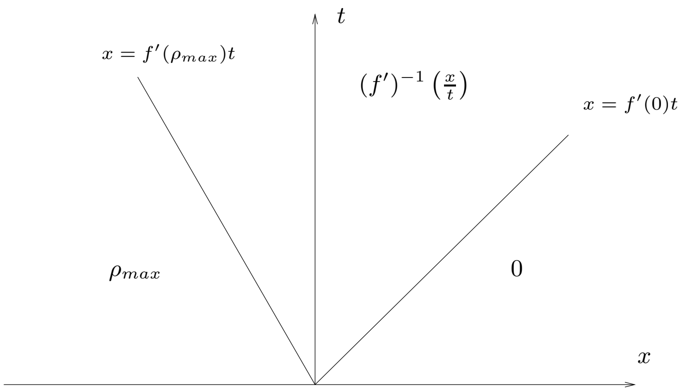


Fig. 1.2. The evolution in the case of a traffic light, when the red light turns green.

The same model describes well the phenomenon of queue formation when a red light starts. Assume now that the initial datum is given by $\rho_0(x) \equiv \sigma$. This typical datum corresponds to a situation in which the traffic light is green, so that cars are flowing through. Assume that the red starts at time $t = 0$, then it is the same as imposing the flux to be zero at $x = 0$. This can be mathematically described by considering the road split in two parts and assign the flux to be zero at $x = 0$ (same as giving boundary data). In this case the solution is given by:

$$\rho(t,x) = \begin{cases} \sigma, & \text{if } x \le -v_{max}\,\sigma\,t, \\ \rho_{max}, & \text{if } -v_{max}\,\sigma\,t < x \le 0, \\ 0, & \text{if } 0 < x < v_{max}\,\sigma\,t, \\ \sigma, & \text{if } x > v_{max}\,\sigma\,t; \end{cases}$$

see figure 1.3. Therefore, for a fixed time $t > 0$, a queue is forming at the light and cars are piling up at the point $-v_{max}\,\sigma\,t$; in front of the light there is an empty region up to the point $v_{max}\,\sigma\,t$.

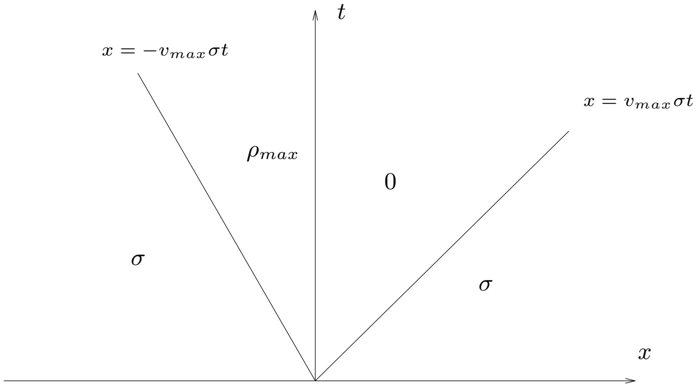


Fig. 1.3. The evolution in the case of a traffic light turning red.

Let us now pass to treat the case of a general network, formed by a finite collection of roads that meet at some junctions. Due to finite speed of waves in solutions to (1.0.1), it is enough to assign the dynamics at each junction separately to obtain an evolution on the whole network. To illustrate the main difficulties, it is enough to focus on a simple junction with one incoming and two outgoing roads. We consider the particular situation in which the incoming road is occupied by cars with maximum density, while the outgoing roads are empty, see Figure 1.4. Indicating by $\rho_{0,i}$ the initial datum on road

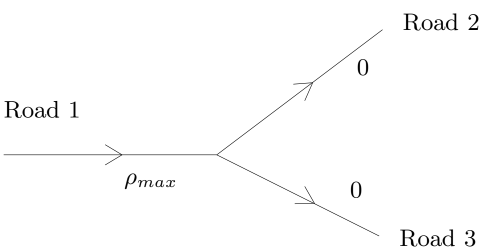


Fig. 1.4. A particular situation in a junction with 1 incoming road and 2 outgoing roads.

i, $i = 1, \ldots, 3$, in formulas we have:

$$\rho_{0,1}(x) = \rho_{max}, \quad \rho_{0,2}(x) = \rho_{0,3}(x) = 0. \tag{1.0.4}$$

There are two extreme behaviors of cars one may expect. Namely: all cars flow towards the first outgoing road or all cars flow towards the second outgoing road. These two behaviors in fact define two different solutions on the network ρ and $\tilde{\rho}$:

$$\rho_1(t,x) = \begin{cases} \rho_{max}, & \text{if } x < f'(\rho_{max})t, \\ (f')^{-1}(\frac{x}{t}), & \text{if } f'(\rho_{max})t < x < 0, \end{cases}$$

$$\rho_2(t,x) = \begin{cases} (f')^{-1}(\frac{x}{t}), & \text{if } 0 \leqslant x < f'(0)t, \\ 0, & \text{if } x > f'(0)t, \end{cases} \qquad \rho_3(t,x) = 0,$$

while $\tilde{\rho}_1 = \rho_1$, $\tilde{\rho}_2 = \rho_3$ and $\tilde{\rho}_3 = \rho_2$.
Notice that both solutions conserve cars quantity through the junction and this can be expressed as:

$$\sum_{\text{incoming roads}} \text{incoming fluxes} = \sum_{\text{outgoing roads}} \text{outgoing fluxes}.$$

Therefore the solely conservation of cars is not sufficient to isolate a unique solution. Moreover, the example shows the necessity of taking into account drivers preferences, in the sense of assigning traffic distribution coefficients, which prescribe the percentage of cars going to each outgoing road. For a general junction with n incoming roads and m outgoing roads, this can be resumed in a general rule:

(A) there exists a traffic distribution matrix

$$A = \begin{pmatrix} \alpha_{n+1,1} & \cdots & \alpha_{n+1,n} \\ \vdots & \vdots & \vdots \\ \alpha_{n+m,1} & \cdots & \alpha_{n+m,n} \end{pmatrix}, \tag{1.0.5}$$

where $0 \leqslant \alpha_{j,i} \leqslant 1$ for every $i \in \{1, \ldots, n\}$ and for every $j \in \{n+1, \ldots, n+m\}$ and

$$\sum_{j=n+1}^{n+m} \alpha_{j,i} = 1 \tag{1.0.6}$$

for every $i \in \{1, \ldots, n\}$. The coefficient $\alpha_{j,i}$ gives the percentage of cars flowing from the i-th incoming road to the j-th outgoing one.

If we denote by f_i and f^j, respectively, the fluxes on the i-th incoming road and on the j-th outgoing one, the rule (A) can be mathematically expressed by the formula

$$\begin{pmatrix} f^1 \\ \vdots \\ f^m \end{pmatrix} = A \cdot \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix}. \tag{1.0.7}$$

First notice that rule (A) is in agreement with conservation of cars through the junction. In fact from (1.0.6):

$$\sum_j f^j = \sum_j \sum_i \alpha_{j,i} f_i = \sum_i \sum_j \alpha_{j,i} f_i = \sum_i f_i.$$

One may expect that rule (A) is sufficient to describe in a unique fashion the dynamics at junctions. Unfortunately this is not the case!
Consider again a junction with one incoming and two outgoing roads and the initial data (1.0.4). A solution satisfying rule (A) is simply given by the functions

$$\rho_1(t,x) = \rho_{max}, \quad \rho_2(t,x) = \rho_3(t,x) = 0,$$

i.e. cars do not cross the junction. Notice that this is clearly a solution to (1.0.1) on each road, in fact all derivatives vanish, and it conserves the number of cars through the junction, since all fluxes vanish. Moreover, for every traffic distribution matrix A, this solution always respects rule (A)! Indeed, given any distribution matrix $A = (\alpha_{j,i})$, equation (1.0.7) reads simply $0 = 0$.
The modelling counterpart of such phenomenon is the fact that the attitude of drivers to cross the junction is not detected by rule (A). The solution above captures the situation in which, for some reason, no driver wants to cross the junction. On the contrary, it is reasonable to assume that the will of drivers is that of reaching their final destination as fast as possible. We thus fix another rule

(B) The number of cars passing the junction is the maximum possible (respecting rule (A)).

Notice that the previous rule is equivalent to maximize the fluxes in incoming roads, i.e. to maximize the functional

$$\sum_{i=1}^{n} f_i.$$

We show that, using rules (A) and (B), one can isolate a unique solution on networks, in case $m \geq n$ (and under some generic assumption). Moreover, in case of a single incoming road, it is easy to check that rule (B) is equivalent to maximize the average velocity on the incoming roads.

One can also treat junctions where the number of incoming roads is greater than the number of outgoing ones. But, in this case, if not all cars can go through the junction then there should be a yielding rule between incoming roads. This corresponds to fix *right of way* parameters, which permit to find a unique solution. More precisely, the i–th parameter indicates the percentage, among cars passing through the junction, coming from the i–th incoming road. The details about the mentioned rules are showed in Section 5.2.

The book develops a complete theory for car traffic, including source-destination models and traffic regulation problems. Also it contains some results for telecommunication networks. We illustrate below the content of each chapter.

1.1 Book Chapters

This book is organized as follows. Chapter 2 deals with hyperbolic systems of conservation laws. We introduce the basic definitions and give the basic tool to prove existence and uniqueness of solutions.

In Chapter 3, some fluidodynamic macroscopic models for traffic on a single road are presented. First we describe in detail the first order model of Lighthill, Whitham and Richards (LWR model). Then we pass to the second order models (i.e. systems of two equations) proposed by Payne and Whitham and by Aw and Rascle. Various other more complex models are included, e.g. multi-lane and multi-population.

Chapter 4 is devoted to the study of a general network, composed by a finite number of edges and vertices. The general approach is presented: Riemann problems at junctions and wave-front tracking algorithms. In particular, we give the definition of Riemann solver at a junction.

Chapter 5 is also focused on road networks, where on each road the scalar LWR model determines the evolution of car traffic. At each junction, we construct the solution satisfying rules (A) and (B) (or with the right of way parameters.) Existence of a solution to Cauchy problems on the whole network is granted. The method, based on the wave-front tracking procedure, described in Chapter 4, makes use of total variation estimates on the fluxes, weak convergence and big waves tracing. The solution happens to be not Lipschitz continuous in the L^1-norm with respect to the initial condition.

In Chapter 6, the Aw-Rascle model is put on a road network. At each junction, we consider again rules (A) and (B), but, in this case, they are not sufficient for uniqueness of solutions. Hence additional rules are in order and we propose three alternative ones. Then stability properties of solutions at junctions are studied for each additional rule. Finally, a solution for the Cauchy problem at a single junction is obtained for initial data, which are small perturbations of stable equilibria.

Chapter 7 deals with an extended model of network, containing sources and destinations. The situation on each road is no more described just by the car density, but also by the traffic types, distinguished on the base of sources and destinations. The resulting model is more complicated than that of Chapter 5, and we consider several equations describing the evolution of traffic-type functions. Using a new Riemann solver at junctions, existence of solutions is proved for perturbations of network equilibria.

In Chapter 8, a typical traffic regulation problem is discussed: when constructing a junction, with some traffic flux expected, is it preferable a traffic light or a circle? Using the model of Chapter 5, we assume that drivers arriving at the junction distribute on the outgoing roads according to some expected coefficients. A comparison between the light and the circle is developed showing the behavior of solutions and, in particular, detecting the situation of stuck traffic.

Then we pass to consider the flow of information on a telecommunication network encoded in packets, in Chapter 9. The analogy with fluids comes from considering packets as particles. Our idea is to look at the network at an intermediate time scale so that packets transmission happens at a faster level but the equilibria of the whole network are reached only as asymptotic. This permits to construct a model relying on a macroscopic description. A new Riemann solver is introduced, to better mimic a router policy, getting a more stable situation in which solutions depend in a Lipschitz fashion from initial data.

Chapter 10 deals with some numerical algorithms to simulate the behavior of the urban traffic flow. We focus on the Lighthill-Whitham-Richards model on each road network and, at junctions, the Riemann solver proposed in Chapter 5. Some numerical schemes, based on the Godunov and Kinetic schemes, are proposed and tested on some networks.

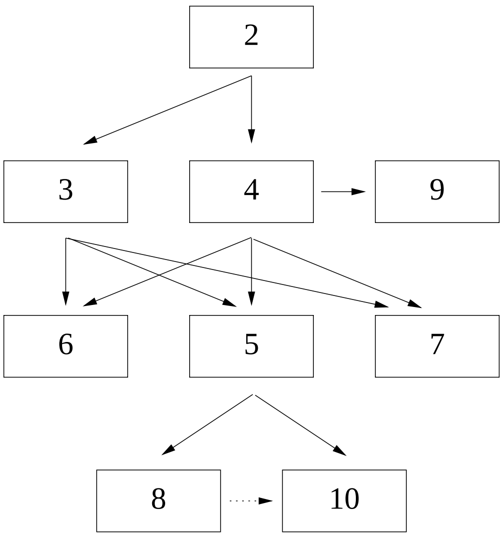


Fig. 1.5. Table of links among book chapters.

A complete course on a fluid-dynamic models for urban traffic can be based on all Chapters excluding Chapter 9. For a short course on urban traffic, the readers can focus on Chapters from 2 to 5 and possibly choose one of Chapters 6, 7, 8 and 10.

A short course on telecommunication networks can be based on Chapters 2, 4, 5 and 9.

2

Conservation Laws

The models for traffic flow we present in this book are based on systems of conservation laws, which are special systems of partial differential equations, where the variables are *conserved quantities*, i.e. quantities which can neither be created nor destroyed.

In this chapter we give some basic preliminaries about systems of conservation laws. A complete theory of hyperbolic systems of conservation laws is not in the aim of the book. Therefore we remand the reader to the books by Bressan [19], by Dafermos [40] and by Serre [98].

2.1 Basic Definitions

A system of conservation laws in one space dimension can be written in the form

$$u_t + f(u)_x = 0, \tag{2.1.1}$$

where $u : [0, +\infty[\times\mathbb{R} \to \mathbb{R}^n$ is the "conserved quantity" and $f : \mathbb{R}^n \to \mathbb{R}^n$ is the flux. Indeed, if we integrate (2.1.1) on an arbitrary space interval $[a, b]$, then

$$\frac{d}{dt}\int_a^b u(t,x)dx = -\int_a^b f(u(t,x))_x dx = f(u(t,a)) - f(u(t,b)),$$

and so the amount of u in any interval $[a, b]$ varies according to the quantity of u entering and exiting at $x = a$ and $x = b$.

We always assume f to be smooth, thus, if u is a smooth function, then (2.1.1) can be rewritten in the quasi linear form

$$u_t + A(u)u_x = 0, \tag{2.1.2}$$

where $A(u)$ is the Jacobian matrix of f at u.

Definition 2.1.1. *The system (2.1.2) is said hyperbolic if, for every $u \in \mathbb{R}^n$, all the eigenvalues of the matrix $A(u)$ are real. Moreover (2.1.2) is said strictly hyperbolic if it is hyperbolic and if, for every $u \in \mathbb{R}^n$, the eigenvalues of the matrix $A(u)$ are all distinct.*

Remark 2.1.2. It is clear that equations (2.1.1) and (2.1.2) are completely equivalent for smooth solutions. If instead u has a jump, the quasilinear equation (2.1.2) is in general not well defined, since there is a product of a discontinuous function $A(u)$ with a Dirac measure.

Example 2.1.3. **The scalar case**. If $n = 1$ so u takes values in $\mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$, then (2.1.1) is a single equation. In this case we say that (2.1.1) is a scalar equation.

Example 2.1.4. **The $n \times n$ system**. If $n > 1$, then (2.1.1) is a system of n equations of conservation laws. Indeed if $u = (u_1, \ldots, u_n)$ and $f = (f_1, \ldots, f_n)$, then (2.1.1) can be written in the form

$$\begin{cases} \partial_t u_1 + \partial_x f_1(u) = 0, \\ \vdots \\ \partial_t u_n + \partial_x f_n(u) = 0. \end{cases}$$

2.2 Weak Solutions

A standard fact for the nonlinear system (2.1.1) is that classical solutions may not exist for some positive time, even if the initial datum is smooth. This can be shown by the method of characteristics. We describe briefly this method for a quasilinear system.

Consider the Cauchy problem

$$\begin{cases} u_t + a(t, x, u)u_x = h(t, x, u), \\ u(0, x) = \bar{u}(x), \end{cases} \tag{2.2.3}$$

and, for every $y \in \mathbb{R}$, the curves $x(t, y)$, $u(t, y)$ solving

$$\begin{cases} \frac{dx}{dt} = a(t, x, u), \\ \frac{du}{dt} = h(t, x, u), \\ x(0, y) = y, \\ u(0, y) = \bar{u}(y). \end{cases} \tag{2.2.4}$$

The curves $t \mapsto x(t, y)$ when $y \in \mathbb{R}$ are called characteristics.

The implicit function theorem implies that the map

$$(t, y) \mapsto (t, x(t, y)) \tag{2.2.5}$$

is locally invertible in a neighborhood of $(0, x_0)$ and so it is possible to consider the map $u(t, x) = u(t, y(t, x))$ where $y(t, x)$ is the inverse of the second component of (2.2.5). It is easy to check that $u(t, x)$ satisfies (2.2.3).

Example 2.2.1. Let us consider the scalar Burgers equation

$$u_t + uu_x = 0,$$

with the initial condition $u(0,x) = u_0(x) = \frac{1}{1+x^2}$. The method of characteristics shows that the solution $u(t,x)$ to this Cauchy problem must be constant along the lines

$$t \mapsto \left(t, x + \frac{t}{1+x^2}\right).$$

For t sufficiently small ($t < \frac{8}{\sqrt{27}}$) these lines do not intersect together and so the solution is classical, but at $t = \frac{8}{\sqrt{27}}$ the characteristics intersect together and a classical solution can not exist for $t \geqslant \frac{8}{\sqrt{27}}$; see Figure 2.1.

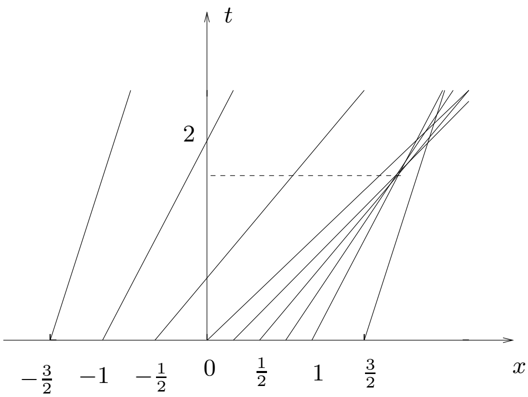


Fig. 2.1. The characteristic for the Burgers equation of Example 2.2.1 in the (t,x)-plane.

Hence we must deal with weak solutions.

Definition 2.2.2. *Fix $u_0 \in L^1_{loc}(\mathbb{R};\mathbb{R}^n)$ and $T > 0$. A function $u : [0,T] \times \mathbb{R} \to \mathbb{R}^n$ is a weak solution to the Cauchy problem*

$$\begin{cases} u_t + f(u)_x = 0, \\ u(0,x) = u_0(x), \end{cases} \tag{2.2.6}$$

if u is continuous as a function from $[0,T]$ into L^1_{loc} and if, for every C^1 function ψ with compact support contained in the set $]-\infty, T[\times\mathbb{R}$, it holds

$$\int_0^T \int_{\mathbb{R}} \{u \cdot \psi_t + f(u) \cdot \psi_x\}\, dxdt + \int_{\mathbb{R}} u_0(x) \cdot \psi(0,x) dx = 0. \tag{2.2.7}$$

Remark 2.2.3. Notice that a weak solution u to (2.2.6) satisfies

$$u(0,x) = u_0(x) \qquad \text{for a.e. } x \in \mathbb{R}.$$

This is a consequence of the fact that u is continuous as a function from $[0,T]$ to L^1_{loc} and of equation (2.2.7).

Weak solutions may develop discontinuities in finite time. We introduce some notations to treat such discontinuities.

Definition 2.2.4. *A function $u = u(t,x)$ has an approximate jump discontinuity at the point (τ,ξ) if there exist vectors $u^-, u^+ \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$ such that*

$$\lim_{r\to 0^+} \frac{1}{r^2}\int_{-r}^{r}\int_{-r}^{r} |u(\tau+t,\xi+x) - U(t,x)|\, dxdt = 0,$$

where

$$U(t,x) := \begin{cases} u^-, & \text{if } x < \lambda t, \\ u^+, & \text{if } x > \lambda t. \end{cases} \tag{2.2.8}$$

The function U is called a shock travelling wave.

The following theorem holds.

Theorem 2.2.5. *Consider a bounded weak solution u to (2.1.1) with an approximate jump discontinuity at (τ,ξ). Then*

$$\lambda(u^+ - u^-) = f(u^+) - f(u^-). \tag{2.2.9}$$

Proof. First suppose that u is equal to (2.2.8). Let Ω be a subset of the (t,x)-plane and $\psi : \Omega \to \mathbb{R}^n$ be a C^1 function with compact support $K \subseteq \Omega$. Consider the vector field

$$g(t,x) := (u(t,x)\cdot\psi(t,x), f(u(t,x))\cdot\psi(t,x))$$

and the sets $K^+ := K \cap \{x > \lambda t\}$, $K^- := K \cap \{x < \lambda t\}$, see Figure 2.2. The outward normal to the boundary of $K^\pm$ is given respectively by

$$\mathbf{n}_+ = -\mathbf{n}_- = \frac{1}{\sqrt{\lambda^2+1}}(\lambda,-1),$$

thus applying the divergence theorem to g on $K^\pm$, we deduce

$$\int_{K^\pm} \mathrm{div}(g)\, dxdt = \int_{\partial K^\pm} \mathbf{n}_\pm \cdot g = \frac{1}{\sqrt{\lambda^2+1}} \int \left[\pm\lambda u\cdot\psi(t,\lambda t\pm) \mp f\cdot\psi(t,\lambda t\pm)\right] dt.$$

Since u is weak solution we get:

$$0 = \int_K \mathrm{div}(g)\, dxdt = \int_{K^+} \mathrm{div}(g)\, dxdt + \int_{K^-} \mathrm{div}(g)\, dxdt,$$

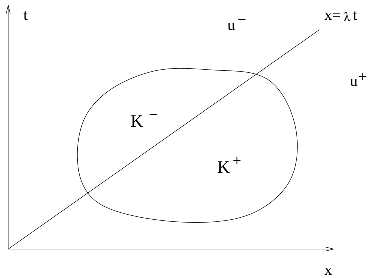


Fig. 2.2. Region where the divergence theorem is applied.

hence

$$0 = \int \left\{ \lambda \left[u^+ - u^- \right] - \left[f(u^+) - f(u^-) \right] \right\} \cdot \psi(t, \lambda t) dt.$$

The arbitrariness of the function ψ implies

$$\lambda \left[u^+ - u^- \right] = f(u^+) - f(u^-).$$

Consider now a general weak solution u to (2.1.1) with an approximate jump discontinuity at (τ, ξ). For every $\eta > 0$, define

$$u^\eta(t, x) := u(\tau + \eta t, \xi + \eta x).$$

We have

$$\begin{aligned} \int_{-r}^{r} \int_{-r}^{r} |u^\eta(t,x) - U(t,x)|\, dxdt &= \int_{-r}^{r} \int_{-r}^{r} |u(\tau + \eta t, \xi + \eta x) - U(t,x)|\, dxdt \\ &= \frac{1}{\eta^2} \int_{-\eta r}^{\eta r} \int_{-\eta r}^{\eta r} \left| u(\tau + t, \xi + x) - U\left(\frac{t}{\eta}, \frac{x}{\eta} \right) \right| dxdt \\ &= r^2 \frac{1}{(\eta r)^2} \int_{-\eta r}^{\eta r} \int_{-\eta r}^{\eta r} |u(\tau + t, \xi + x) - U(t,x)|\, dxdt, \end{aligned}$$

where we used the definitions of u^η and of U. From Definition 2.2.4, the last term goes to 0 as $\eta \to 0$ and so $u^\eta \to U$ in L^1_{loc} as $\eta \to 0$. Moreover $f(u^\eta) \to f(U)$ in L^1_{loc} as $\eta \to 0$, since u is bounded and f is Lipschitz continuous on bounded sets. Therefore U is a weak solution to (2.1.1) and

$$\lambda \left[u^+ - u^- \right] = f(u^+) - f(u^-)$$

by the previous analysis. □

Equation (2.2.9), called Rankine-Hugoniot condition, gives a condition on discontinuities of weak solutions of (2.1.1) relating the right and left states with the "speed" λ of the "shock". In the scalar case (2.2.9) is a single equation and, for arbitrary $u^- \neq u^+$, we have

$$\lambda = \frac{f(u^+) - f(u^-)}{u^+ - u^-}.$$

For a $n \times n$ system of conservation laws, (2.2.9) is a system of n scalar equations.

Example 2.2.6. Consider the Burgers equation

$$u_t + \left(\frac{u^2}{2}\right)_x = 0 \tag{2.2.10}$$

with the initial condition

$$u_0(x) = \begin{cases} 1 - |x|, & \text{if } x \in [-1, 1], \\ 0, & \text{otherwise.} \end{cases} \tag{2.2.11}$$

The characteristics in this case are drawn in Figure 2.3.

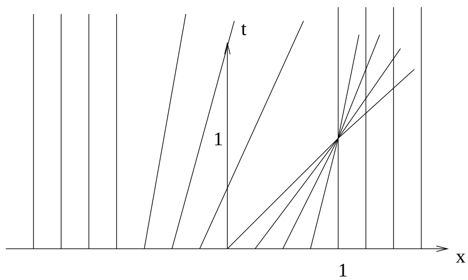


Fig. 2.3. Superposition of characteristic curves for a Burgers equation.

Therefore for $0 \leqslant t < 1$, the function

$$u(t,x) = \begin{cases} \frac{x+1}{t+1}, & \text{if } -1 \leqslant x < t, \\ \frac{1-x}{1-t}, & \text{if } t < x \leqslant < 1, \\ 0, & \text{otherwise,} \end{cases}$$

is a classical solution to (2.2.10). The Rankine-Hugoniot condition in the case of Burgers equation reduces to

$$\lambda = \frac{\left[\frac{(u^+)^2}{2}\right] - \left[\frac{(u^-)^2}{2}\right]}{u^+ - u^-} = \frac{u^+ + u^-}{2}.$$

If $t \geqslant 1$, then the function

$$u(t,x) = \begin{cases} \frac{x+1}{t+1}, & \text{if } -1 \leqslant x \leqslant -1 + \sqrt{2+2t}, \\ 0, & \text{otherwise,} \end{cases}$$

satisfies the Rankine-Hugoniot condition at each point of discontinuity and so a weak solution to the Cauchy problem (2.2.10)-(2.2.11) exists for each positive time; see Figure 2.4.

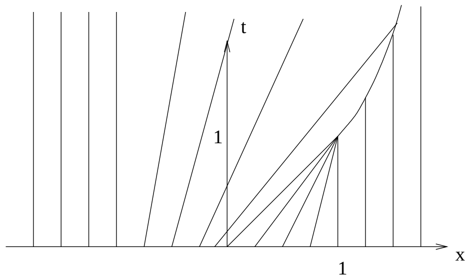


Fig. 2.4. Solution to Burgers equation of the example.

Example 2.2.7. Let u_0 be the function defined by

$$u_0(x) := \begin{cases} 1, & \text{if } x \geqslant 0, \\ 0, & \text{if } x < 0. \end{cases}$$

For every $0 < \alpha < 1$, the function $u_\alpha : [0, +\infty[\times \mathbb{R} \to \mathbb{R}$ defined by

$$u_\alpha(t,x) := \begin{cases} 0, & \text{if } x < \frac{\alpha t}{2}, \\ \alpha, & \text{if } \frac{\alpha t}{2} \leqslant x < \frac{(1+\alpha)t}{2}, \\ 1, & \text{if } x \geqslant \frac{(1+\alpha)t}{2}, \end{cases}$$

is a weak solution to the Burgers equation (2.2.10); see Figure 2.5.

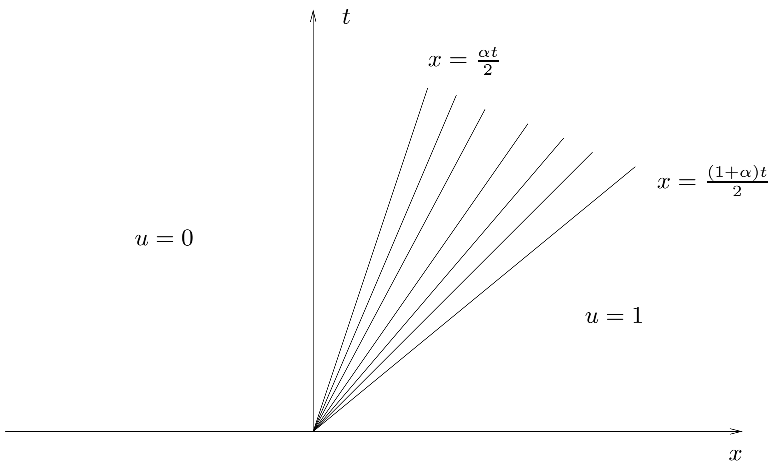


Fig. 2.5. A solution u_α.

Example 2.2.7 shows that the definition of weak solution does not guarantee uniqueness. Therefore the notion of weak solution must be supplemented with admissibility conditions, motivated by physical considerations.

2.3 Entropy Admissible Solutions

A first admissibility criterion, coming from physical considerations, see Dafermos [40], is that of the entropy-admissibility condition.

Definition 2.3.1. *A C^1 function $\eta : \mathbb{R}^n \to \mathbb{R}$ is an entropy for (2.1.1) if it is convex and there exists a C^1 function $q : \mathbb{R}^n \to \mathbb{R}$ such that*

$$D\eta(u) \cdot Df(u) = Dq(u) \tag{2.3.12}$$

for every $u \in \mathbb{R}^n$. The function q is said an entropy flux for η. The pair (η, q) is said entropy–entropy flux pair for (2.1.1).

Definition 2.3.2. *A weak solution $u = u(t, x)$ to the Cauchy problem*

$$\begin{cases} u_t + f(u)_x = 0, \\ u(0, x) = u_0(x), \end{cases} \tag{2.3.13}$$

is said entropy admissible if, for every C^1 function $\varphi \geqslant 0$ with compact support in $[0, T[\times\mathbb{R}$ and for every entropy–entropy flux pair (η, q), it holds

$$\int_0^T \int_{\mathbb{R}} \{\eta(u)\varphi_t + q(u)\varphi_x\}\, dxdt \geqslant 0. \tag{2.3.14}$$

We consider now an entropy admissible solution u and a sequence of entropy–entropy flux pairs (η_ν, q_ν) such that $\eta_\nu \to \eta$ and $q_\nu \to q$ locally uniformly in $u \in \mathbb{R}^n$. If $\varphi \geqslant 0$ is a C^1 function with compact support in $[0, T[\times\mathbb{R}$, then

$$\int_0^T \int_{\mathbb{R}} \{\eta_\nu(u)\varphi_t + q_\nu(u)\varphi_x\}\, dxdt \geqslant 0 \tag{2.3.15}$$

for every $\nu \in \mathbb{N}$. Passing to the limit as $\nu \to +\infty$ in (2.3.15), we obtain that

$$\int_0^T \int_{\mathbb{R}} \{\eta(u)\varphi_t + q(u)\varphi_x\}\, dxdt \geqslant 0. \tag{2.3.16}$$

This allows us to call a C^0 function η an entropy if there exists a sequence of entropies η_ν converging to η locally uniformly. Moreover a C^0 function q is a corresponding entropy flux if there exists a sequence q_ν, entropy flux of η_ν, converging to q locally uniformly.

2.3.1 Scalar Case

Let us consider the scalar Cauchy problem

$$\begin{cases} u_t + f(u)_x = 0, \\ u(0, x) = u_0(x), \end{cases} \tag{2.3.17}$$

where $f : \mathbb{R} \to \mathbb{R}$ is a C^1 function. In this case the relation between C^1 entropy and entropy flux takes the form

$$\eta'(u)f'(u) = q'(u). \tag{2.3.18}$$

Therefore if we take a C^1 entropy η, every corresponding entropy flux q has the expression

$$q(u) = \int_{u_0}^{u} \eta'(s)f'(s)ds,$$

where u_0 is an arbitrary element of $\mathbb{R}$.

Definition 2.3.3. *A weak solution $u = u(t,x)$ to the scalar Cauchy problem (2.3.17) satisfies the Kruzkov entropy admissibility condition if*

$$\int_0^T \int_{\mathbb{R}} \{|u-k|\,\varphi_t + sgn\,(u-k)\,(f(u)-f(k))\,\varphi_x\}\,dxdt \geqslant 0$$

for every $k \in \mathbb{R}$ and every C^1 function $\varphi \geqslant 0$ with compact support in $[0,T[\times\mathbb{R}$.

We have the following theorem.

Theorem 2.3.4. *Let $u = u(t,x)$ be a piecewise C^1 solution to the scalar equation (2.3.17). Then u satisfies the Kruzkov entropy admissible condition if and only if along every line of jump $x = \xi(t)$ the following condition holds. For every $\alpha \in [0,1]$*

$$\begin{cases} f(\alpha u^+ + (1-\alpha)u^-) \geqslant \alpha f(u^+) + (1-\alpha)f(u^-), & \text{if } u^- < u^+, \\ f(\alpha u^+ + (1-\alpha)u^-) \leqslant \alpha f(u^+) + (1-\alpha)f(u^-), & \text{if } u^- > u^+, \end{cases} \tag{2.3.19}$$

where $u^- := u(t,\xi(t)-)$ and $u^+ := u(t,\xi(t)+)$.

For a proof of this theorem see [19]. Equation (2.3.19) implies that, if $u^- < u^+$, then the graph of f remains above the segment connecting $(u^-, f(u^-))$ to $(u^+, f(u^+))$ (see Figure 2.6), while if $u^- > u^+$, then the graph of f remains below the segment connecting $(u^-, f(u^-))$ to $(u^+, f(u^+))$ (see Figure 2.7).

2.4 Riemann Problem

This section describes the entropy admissible solutions to a Riemann problem, i.e. a Cauchy problem with Heaviside initial data. Let $\Omega \subseteq \mathbb{R}^n$ be an open set, let $f : \Omega \to \mathbb{R}^n$ a smooth flux and consider the system of conservation laws

$$u_t + f(u)_x = 0, \tag{2.4.20}$$

which we suppose to be strictly hyperbolic.

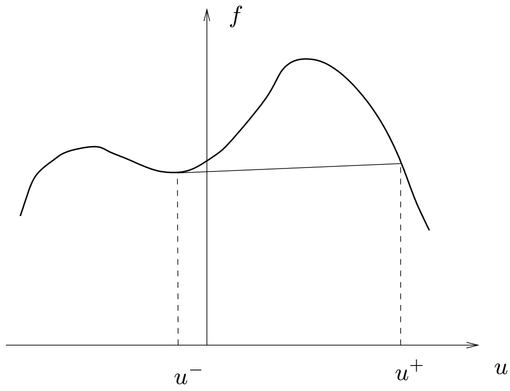


Fig. 2.6. The condition 2.3.19 in the case $u^- < u^+$.

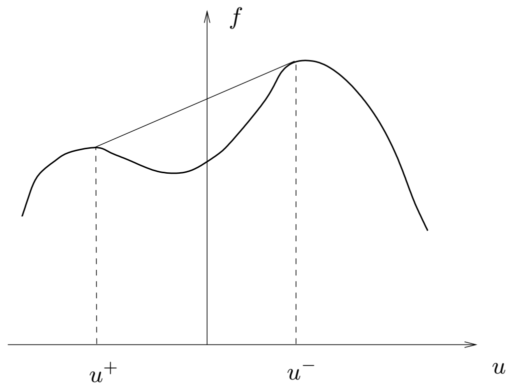


Fig. 2.7. The condition 2.3.19 in the case $u^- > u^+$.

Definition 2.4.1. *A Riemann problem for the system (2.4.20) is the Cauchy problem for the initial datum*

$$u_0(x) := \begin{cases} u^-, & \textit{if } x < 0, \\ u^+, & \textit{if } x > 0, \end{cases} \tag{2.4.21}$$

where $u^-, u^+ \in \Omega$.

Remark 2.4.2. As shown in Section 2.6, the solution of Riemann problems is the key step to solve Cauchy problems. In fact to prove existence we use the wave-front tracking method, that, roughly speaking, consists in the following:

1. approximate the initial condition with piecewise constant solutions;
2. at every point of discontinuity solve the corresponding Riemann problem;
3. approximate the exact solution to Riemann problems with piecewise constant functions and piece them together to get a function defined until two wave fronts interact together;

4. repeat inductively the previous construction starting from the interaction time;
5. prove that the functions so constructed converge to a limit function and prove that this limit function is an entropy admissible solution.

As before we denote by $A(u)$ the Jacobian matrix of the flux f and with $\lambda_1(u) < \cdots < \lambda_n(u)$ the n eigenvalues of the matrix $A(u)$. Let $\{r_1(u), \dots, r_n(u)\}$, $\{l_1(u), \dots, l_n(u)\}$ be, respectively, bases of right and left eigenvectors such that

1. $|r_i(u)| \equiv 1$ for every $u \in \Omega$ and $i \in \{1, \cdots, n\}$;
2. $l_i \cdot r_j = \delta_{ij}$ for every $i, j \in \{1, \cdots, n\}$, where δ_{ij} denotes the Kronecker symbol, that is

$$\delta_{ij} := \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}$$

We introduce the following notation. If $i \in \{1, \cdots, n\}$, then

$$r_i \bullet \lambda_j(u) := \lim_{\varepsilon \to 0} \frac{\lambda_j(u + \varepsilon r_i(u)) - \lambda_j(u)}{\varepsilon},$$

that is the directional derivative of $\lambda_j(u)$ in the direction of $r_i(u)$.

Definition 2.4.3. *We say that the i-characteristic field ($i \in \{1, \cdots, n\}$) is genuinely nonlinear if*

$$r_i \bullet \lambda_i(u) \neq 0 \qquad \forall u \in \Omega.$$

We say that the i-characteristic field ($i \in \{1, \cdots, n\}$) is linearly degenerate if

$$r_i \bullet \lambda_i(u) = 0 \qquad \forall u \in \Omega.$$

If the i-th characteristic field is genuinely nonlinear, then, for simplicity, we assume that $r_i \bullet \lambda_i(u) > 0$ for every $u \in \Omega$.

We consider three cases.

1. Centered rarefaction waves. For $u^- \in \Omega$, $i \in \{1, \cdots, n\}$ and $\sigma > 0$, we denote by $R_i(\sigma)(u^-)$ the solution to

$$\begin{cases} \frac{du}{d\sigma} = r_i(u), \\ u(0) = u^-. \end{cases} \tag{2.4.22}$$

Let $\bar{\sigma} > 0$. Define $u^+ = R_i(\bar{\sigma})(u^-)$ for some $i \in \{1, \cdots, n\}$. If the i-th characteristic field is genuinely nonlinear, then the function

$$u(t, x) := \begin{cases} u^-, & \text{if } x < \lambda_i(u^-)t, \\ R_i(\sigma)(u^-), & \text{if } x = \lambda_i(R_i(\sigma)(u^-))t,\ \sigma \in [0, \bar{\sigma}], \\ u^+, & \text{if } x > \lambda_i(u^+)t, \end{cases} \tag{2.4.23}$$

is an entropy admissible solution to the Riemann problem

$$\begin{cases} u_t + f(u)_x = 0, \\ u(0,x) = u_0(x), \end{cases}$$

with u_0 defined in (2.4.21). The function $u(t,x)$ is called a centered rarefaction wave.

Remark 2.4.4. Notice that, to construct function (2.4.23), $\bar{\sigma}$ must be positive.

2. Shock waves. Fix $u^- \in \Omega$ and $i \in \{1, \cdots, n\}$. For some $\sigma_0 > 0$, there exist smooth functions $S_i(u_-) = S_i : [-\sigma_0, \sigma_0] \to \Omega$ and $\lambda_i : [-\sigma_0, \sigma_0] \to \mathbb{R}$ such that:
   a) $f(S_i(\sigma)) - f(u^-) = \lambda_i(\sigma)(S_i(\sigma) - u^-)$ for every $\sigma \in [-\sigma_0, \sigma_0]$;
   b) $\left|\frac{dS_i}{d\sigma}\right| \equiv 1$;
   c) $S_i(0) = u^-$, $\lambda_i(0) = \lambda_i(u^-)$;
   d) $\frac{dS_i(\sigma)}{d\sigma}|_{\sigma=0} = r_i(u^-)$;
   e) $\frac{d\lambda_i(\sigma)}{d\sigma}|_{\sigma=0} = \frac{1}{2} r_i \bullet \lambda_i(u^-)$;
   f) $\frac{d^2 S_i(\sigma)}{d\sigma^2}|_{\sigma=0} = r_i \bullet r_i(u^-)$.

   Let $\bar{\sigma} < 0$. Define $u^+ = S_i(\bar{\sigma})$. If the i-th characteristic field is genuinely nonlinear, then the function

$$u(t,x) := \begin{cases} u^-, & \text{if } x < \lambda_i(\bar{\sigma})t, \\ u^+, & \text{if } x > \lambda_i(\bar{\sigma})t, \end{cases} \tag{2.4.24}$$

   is an entropy admissible solution to the Riemann problem

$$\begin{cases} u_t + f(u)_x = 0, \\ u(0,x) = u_0(x), \end{cases}$$

   with u_0 defined in (2.4.21). The function $u(t,x)$ is called a shock wave.

   Remark 2.4.5. If we consider $\bar{\sigma} > 0$, then (2.4.24) is again a weak solution, but it does not satisfy the entropy condition.

3. Contact discontinuities. Fix $u^- \in \Omega$, $i \in \{1, \cdots, n\}$ and $\bar{\sigma} \in [-\sigma_0, \sigma_0]$. Define $u^+ = S_i(\bar{\sigma})$. If the i-th characteristic field is linearly degenerate, then the function

$$u(t,x) := \begin{cases} u^-, & \text{if } x < \lambda_i(u^-)t, \\ u^+, & \text{if } x > \lambda_i(u^-)t, \end{cases} \tag{2.4.25}$$

   is an entropy admissible solution to the Riemann problem

$$\begin{cases} u_t + f(u)_x = 0, \\ u(0,x) = u_0(x), \end{cases}$$

   with u_0 defined in (2.4.21). The function $u(t,x)$ is called a contact discontinuity.

Remark 2.4.6. If the i-th characteristic field is linearly degenerate, then

$$\lambda_i(u^-) = \lambda_i(u^+) = \lambda_i(\sigma)$$

for every $\sigma \in [-\sigma_0, \sigma_0]$.

Definition 2.4.7. *The waves defined in* (2.4.23), (2.4.24) *and* (2.4.25) *are called waves of the i-th family.*

For each $\sigma \in \mathbb{R}$ and $i \in \{1, \ldots, n\}$, let us consider the function

$$\psi_i(\sigma)(u_0) := \begin{cases} R_i(\sigma)(u_0), & \text{if } \sigma \geqslant 0, \\ S_i(\sigma)(u_0), & \text{if } \sigma < 0, \end{cases} \tag{2.4.26}$$

where $u_0 \in \Omega$. The value σ is called the strength of the wave of the i-th family, connecting u_0 to $\psi_i(\sigma)(u_0)$. It follows that $\psi_i(\cdot)(u_0)$ is a smooth function. Moreover let us consider the composite function

$$\Psi(\sigma_1, \ldots, \sigma_n)(u^-) := \psi_n(\sigma_n) \circ \cdots \circ \psi_1(\sigma_1)(u^-), \tag{2.4.27}$$

where $u^- \in \Omega$ and $(\sigma_1, \ldots, \sigma_n)$ belongs to a neighborhood of 0 in $\mathbb{R}^n$. It is not difficult to calculate the Jacobian matrix of the function Ψ and to prove that it is invertible in a neighborhood of $(0, \ldots, 0)$. Hence we can apply the Implicit Function Theorem and prove the following result.

Theorem 2.4.8. *For every compact set $K \subseteq \Omega$, there exists $\delta > 0$ such that, for every $u^- \in K$ and for every $u^+ \in \Omega$ with $|u^+ - u^-| \leqslant \delta$ there exists a unique $(\sigma_1, \ldots, \sigma_n)$ in a neighborhood of $0 \in \mathbb{R}^n$ satisfying*

$$\Psi(\sigma_1, \ldots, \sigma_n)(u^-) = u^+.$$

Moreover the Riemann problem connecting u^- with u^+ has an entropy admissible solution, constructing by piecing together the solutions of n Riemann problems.

Example 2.4.9. **(The p-system)**. Consider the following 2×2 hyperbolic system:

$$\begin{cases} \partial_t \rho + \partial_x q = 0, \\ \partial_t q + \partial_x \left(\frac{q^2}{\rho} + p(\rho) \right) = 0. \end{cases} \tag{2.4.28}$$

This system describes an isentropic gas in Eulerian coordinates: $\rho > 0$ is the density of the gas and q is the linear momentum density, i.e. $q = \rho v$ where v is the speed of the gas. The function p is the pressure and depends only on the density ρ. Assume that p is of class C^2 and

$$p(\rho) > 0, \quad p'(\rho) > 0, \quad p''(\rho) \geqslant 0$$

for every $\rho > 0$. A typical example is the γ-pressure law $p(\rho) = k\rho^\gamma$ for $k > 0$ and $\gamma \geq 1$. Consider the coordinate $U = (\rho, q)^T$. Thus (2.4.28) can be rewritten in the form

$$U_t + f(U)_x = 0,$$

where the flux function is

$$f(U) = \begin{pmatrix} q \\ \frac{q^2}{\rho} + p(\rho) \end{pmatrix}.$$

The Jacobian matrix for the flux f is

$$A(U) = \begin{pmatrix} 0 & 1 \\ -\frac{q^2}{\rho^2} + p'(\rho) & 2\frac{q}{\rho} \end{pmatrix},$$

which has the distinct eigenvalues

$$\lambda_1 = \frac{q}{\rho} - \sqrt{p'(\rho)} < \lambda_2 = \frac{q}{\rho} + \sqrt{p'(\rho)},$$

and the corresponding right eigenfunctions

$$r_1 = \begin{pmatrix} \rho, \\ q - \rho\sqrt{p'(\rho)} \end{pmatrix}, \quad r_2 = \begin{pmatrix} \rho, \\ q + \rho\sqrt{p'(\rho)} \end{pmatrix}.$$

This implies that the system (2.4.28) is strictly hyperbolic. Moreover

$$r_1 \bullet \lambda_1 = -\sqrt{p'(\rho)} - \rho\frac{p''(\rho)}{2\sqrt{p'(\rho)}}, \quad r_2 \bullet \lambda_2 = \sqrt{p'(\rho)} + \rho\frac{p''(\rho)}{2\sqrt{p'(\rho)}}$$

and so the characteristic fields are genuinely nonlinear.

Let us consider the Riemann problem for (2.4.28) with initial data

$$U(0,x) = \begin{cases} U^-, & \text{if } x < 0, \\ U^+, & \text{if } x > 0. \end{cases}$$

The equations $\frac{d}{d\sigma}U = r_i(U)$ gives the following rarefaction curves starting at U^-.

$$R_1 = \left\{ (\rho, q) : q = \frac{\rho q^-}{\rho^-} - \rho\int_{\rho^-}^{\rho} \frac{\sqrt{p'(r)}}{r}dr, \quad \rho \leqslant \rho^- \right\},$$

$$R_2 = \left\{ (\rho, q) : q = \frac{\rho q^-}{\rho^-} + \rho\int_{\rho^-}^{\rho} \frac{\sqrt{p'(r)}}{r}dr, \quad \rho \geqslant \rho^- \right\}.$$

The Rankine-Hugoniot condition, instead, gives the shock curves S_1 and S_2 starting at U^-, which are

$$S_1 = \left\{ (\rho, q) : q = \frac{\rho q^-}{\rho^-} - \sqrt{\frac{\rho}{\rho^-}(\rho - \rho^-)\left(p(\rho) - p(\rho^-)\right)}, \quad \rho \geqslant \rho^- \right\},$$

$$S_2 = \left\{ (\rho, q) : q = \frac{\rho q^-}{\rho^-} - \sqrt{\frac{\rho}{\rho^-}(\rho - \rho^-)\left(p(\rho) - p(\rho^-)\right)}, \quad \rho \leqslant \rho^- \right\}.$$

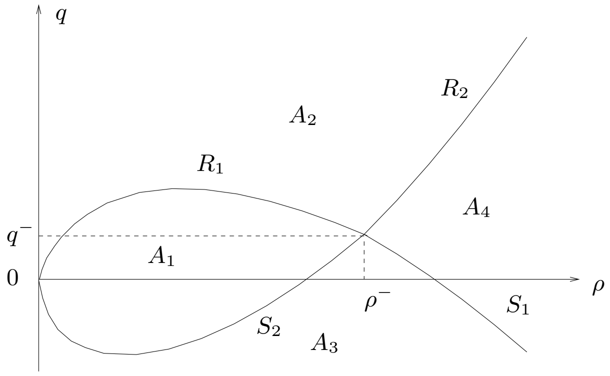


Fig. 2.8. The rarefaction and shock curves.

The situation is described in Figure 2.8. The curves R_i and S_i divide the (ρ, q) plane into four regions A_1, A_2, A_3 and A_4. If U^+ belongs to one of these curves, then the Riemann problem is solved by a single wave. If instead U^+ is sufficiently near to U^- and belongs to one of the regions A_i, then the solution to the Riemann problem is given by two centered waves. More precisely, if $U^+ \in A_1$, then the solution is given by a rarefaction wave of the first family and by a shock wave of the second family. If $U^+ \in A_2$, then the solution is given by two rarefaction waves. If $U^+ \in A_3$, then the solution is given by two shock waves. If $U^+ \in A_4$, then the solution is given by a shock wave of the first family and by a rarefaction wave of the second family.

2.4.1 The Non-Convex Scalar Case

In the scalar case, the construction of solutions to Riemann problems can be done not only in the genuinely nonlinear case, i.e. for convex or concave flux (see Exercise 2.8.4), or linearly degenerate case, i.e. affine flux (see Exercise 2.8.4.)

Consider thus a scalar conservation law:

$$u_t + f(u)_x = 0,$$

with $f : \mathbb{R} \to \mathbb{R}$ smooth. Given (u_-, u_+) the solution to the corresponding Riemann problem is done in the following way.
If $u_- < u_+$ we let $\tilde{f}$ be the largest convex function such that for every $u \in [u_-, u_+]$, it holds:

$$\tilde{f}(u) \leqslant f(u);$$

see Figure 2.9.
If $u_- > u_+$ we let $\tilde{f}$ be the smallest concave function such that for every $u \in [u_+, u_-]$, it holds:

$$\tilde{f}(u) \geqslant f(u);$$

see Figure 2.9.
Then the solution to the Riemann problem with data (u_-, u_+) is the solution for the flux $\tilde{f}$ to the same Riemann problem.

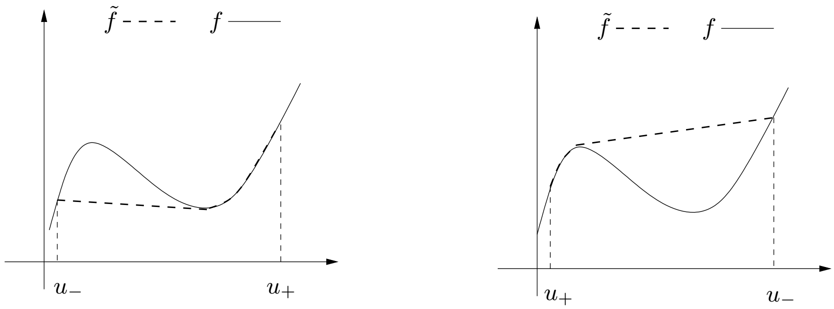


Fig. 2.9. Definition of $\tilde{f}$.

Notice that, in this case, the flux $\tilde{f}$ is in general not strictly convex or concave but may contain some linear part. The solution to the corresponding Riemann problems may contain combinations of rarefactions and shocks. For simplicity, we illustrate only a special case.

Fix the scalar conservation law:

$$u_t + (u^3)_x = 0,$$

and $u_- > 0$.
If $u_+ > u_-$, then $\tilde{f}$ coincides with f and the solution to the corresponding Riemann problem is given by a single rarefaction wave.
If $u_+ < u_-$, then we have to distinguish two cases. First, for every u define $\alpha(u) \leqslant u$ to be the point such that the secant from $(\alpha(u), f(\alpha(u))$ to $(u, f(u))$ is tangent to the graph of $f(u) = u^3$ at $\alpha(u)$; see Figure 2.10.

In formulas:

$$\frac{f(u) - f(\alpha(u))}{u - \alpha(u)} = f'(\alpha(u)),$$

then:

$$\frac{u^3 - \alpha^3(u)}{u - \alpha(u)} = 3\alpha^2(u),$$

and one can easily get two solutions. The trivial one $\alpha(u) = u$ and

$$\alpha(u) = -\frac{u}{2}.$$

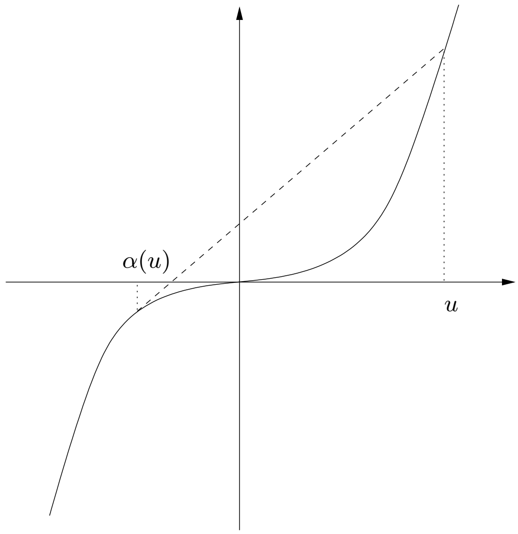


Fig. 2.10. Definition of α.

Now if $u_+ \geqslant \alpha(u_-)$ then again $\tilde{f}$ coincides with f and the solution is given by a single shock. If, on the contrary $u_+ < \alpha(u_-)$, the solution to the Riemann problem is given by the function:

$$u(t,x) = \begin{cases} u_-\,, & \text{if } x < \frac{3}{4}u_-\,t, \\ -\sqrt{\frac{x}{3t}}\,, & \text{if } \frac{3}{4}u_-\,t \leqslant x \leqslant 3(u_+)^2\,t, \\ u_+\,, & \text{if } x > 3(u_+)^2\,t. \end{cases}$$

which is formed by a shock followed by a rarefaction attached to it; see Figure 2.11.

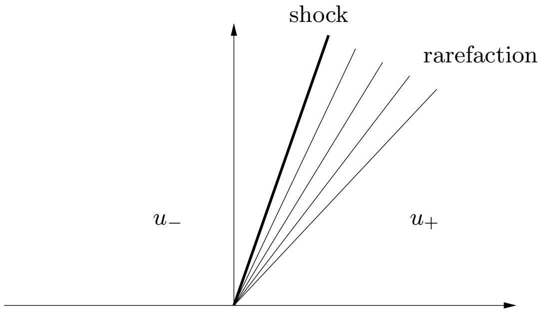


Fig. 2.11. Solution to the Riemann problem with $u_- > 0$ and $u^+ < \alpha(u^-)$.

In the case $u_- < 0$ the construction is symmetric with respect to the case $u_- > 0$, while for $u_- = 0$ the solution is always given by a rarefaction.

2.5 Functions with Bounded Variation

In this section we give some basic facts about functions with bounded variation.

Consider an interval J contained in $\mathbb{R}$ and a function $w : J \to \mathbb{R}$. The total variation of w is defined by

$$\text{Tot.Var.}\, w = \sup \left\{ \sum_{j=1}^{N} |w(x_j) - w(x_{j-1})| \right\}, \tag{2.5.29}$$

where $N \geqslant 1$, the points x_j belong to J for every $j \in \{0, \dots, N\}$ and satisfy $x_0 < x_1 < \cdots < x_N$.

Definition 2.5.1. *We say that the function $w : J \to \mathbb{R}$ has bounded total variation if Tot.Var.$\,w < +\infty$. We denote with $BV(J)$ the set of all real functions $w : J \to \mathbb{R}$ with bounded total variation.*

Notice that the total variation of a function w is a positive number. If w is a function with bounded total variation, then it is clear that w is a bounded function. The converse is false. In fact every non constant periodic and bounded function on $\mathbb{R}$ has total variation equal to $+\infty$. An important property of functions with bounded total variation is the existence of left and right limits for every x of the interior of J.

Lemma 2.5.2. *Let $w : J \to \mathbb{R}$ be a function with bounded total variation and $\bar{x}$ be a point in the interior of J. Then the limits*

$$\lim_{x \to \bar{x}^-} w(x), \qquad \lim_{x \to \bar{x}^+} w(x)$$

exist. Moreover the function w has at most countably many points of discontinuity.

Proof. Fix a point $\bar{x}$ in the interior of J. Consider a strictly increasing sequence $x_n \in J$ converging to $\bar{x}$. We have

$$\sum_{n \in \mathbb{N}, n \geqslant 1} |w(x_n) - w(x_{n+1})| \leqslant \text{Tot.Var.}\, w < +\infty.$$

This implies that the sequence $w(x_n)$ is a Cauchy sequence and thus it converges to some real number w^-. Consider now another strictly increasing sequence $y_n \in J$ converging to $\bar{x}$. As before, the sequence $w(y_n)$ is a Cauchy sequence and thus it converges to some real number $\tilde{w}^-$. We claim that

$w^- = \tilde{w}^-$. In fact, from the sequences x_n and y_n we can construct a unique non-decreasing sequence $\tilde{x}_n$ with the following property. For every $n \in \mathbb{N}$ there exist a unique $n_1 \in \mathbb{N}$ ($n_1 \geqslant n$) and a unique $n_2 \in \mathbb{N}$ ($n_2 \geqslant n$) such that $\tilde{x}_{n_1} = x_n$ and $\tilde{x}_{n_2} = y_n$. We deduce that $\tilde{x}_n$ is a Cauchy sequence; hence it has a unique limit and so $w^- = \tilde{w}^-$. This proves that $\lim_{x\to\bar{x}^-} w(x)$ exists. In the same way it is possible to prove that $\lim_{x\to\bar{x}^+} w(x)$ exists.

Define now, for every $n \in \mathbb{N} \setminus \{0\}$, the set

$$A_n = \left\{ x \in \operatorname{Int} J : |w(x-) - w(x)| + |w(x+) - w(x)| > \frac{1}{n} \right\},$$

where $\operatorname{Int} J$ denotes the interior of J. It is clear that the number of elements of the set A_n can not be more than $n \cdot \text{Tot.Var.}\, w$; hence A_n is a finite set for every $n \geqslant 1$. The set of discontinuities of the function w is the union

$$\bigcup_{n\geqslant 1} A_n,$$

and so it is at most countable. □

The next theorem shows that subsets of $BV(J)$, with uniform bound in total variation, have some compactness properties.

Theorem 2.5.3. ***(Helly)*** *Consider a sequence of functions $w_n : J \to \mathbb{R}^m$. Assume that there exist positive constant C and M such that:*

1. *Tot.Var. $w_n \leqslant C$ for every $n \in \mathbb{N}$;*
2. *$|w_n(x)| \leqslant M$ for every $n \in \mathbb{N}$ and $x \in J$.*

Then there exist a function $w : J \to \mathbb{R}^m$ and a subsequence w_{n_k} such that

1. *$\lim_{k\to+\infty} w_{n_k}(x) = w(x)$ for every $x \in J$;*
2. *Tot.Var. $w \leqslant C$;*
3. *$|w(x)| \leqslant M$ for every $x \in J$.*

Proof. For every $n \in \mathbb{N}$ and $x \in J$, define the function

$$W_n(x) = \sup \left\{ \sum_{j=1}^{N} |w_n(x_j) - w_n(x_{j-1})| \right\}.$$

where the supremum is taken in $N \geqslant 1$, $x_0 \in J$ and $x_0 < x_1 < \cdots < x_N = x$. The value of the function W_n at a certain point $x \in J$ is the total variation of the function w_n until the point x. Moreover we have

$$0 \leqslant W_n(x) \leqslant C, \tag{2.5.30}$$

for every $n \in \mathbb{N}$ and $x \in J$ and

$$|w_n(y) - w_n(x)| \leqslant W_n(y_2) - W_n(y_1), \tag{2.5.31}$$

for every $n \in \mathbb{N}$, $x, y, y_1, y_2 \in J$ and $y_1 \leqslant x \leqslant y \leqslant y_2$. By (2.5.30) and a diagonal procedure, there exist a subsequence W_{n_h} of W_n and a function W such that

$$\lim_{h \to +\infty} W_{n_h}(x) = W(x)$$

for every $x \in J \cap \mathbb{Q}$. Define, for every $n \in \mathbb{N}$,

$$B_n = \left\{ x \in \operatorname{Int} J : \lim_{y \to x^+} W(x) - \lim_{y \to x^-} W(x) \geqslant \frac{1}{n} \right\}.$$

The set B_n is finite and it can contain at most Cn points; hence the set

$$B = \bigcup_{n \in \mathbb{N}} B_n$$

is at most countable. It implies that the function W has at most a countable number of discontinuities.

By hypotheses it is also possible to choose a subsequence n_{h_k} of n_h, which we call for simplicity n_k, such that

$$\lim_{k \to +\infty} w_k(x) = w(x)$$

for every $x \in J \cap (\mathbb{Q} \cup B)$. Indeed we have that the previous limit exists for every $x \in J$. Assume that $\bar{x} \in J \setminus B$. This implies that $\bar{x} \notin B_n$ for every $n \in \mathbb{N}$ and so, for every $n \in \mathbb{N}$, there exist $y_1 < \bar{x} < y_2$, $y_1, y_2 \in \mathbb{Q}$ such that $W(y_2) - W(y_1) < \frac{1}{n}$. We have

$$\begin{aligned} \limsup_{k, \tilde{k} \to +\infty} \left| w_k(\bar{x}) - w_{\tilde{k}}(\bar{x}) \right| &\leqslant 2 \limsup_{k \to +\infty} |w_k(\bar{x}) - w_k(y_1)| \\ &\leqslant 2 \left(W(y_2) - W(y_1) \right) < \frac{2}{n}. \end{aligned}$$

Finally consider points $x_0 < x_1 < \cdots < x_N$ in the set J. We deduce that

$$\sum_{j=1}^{N} |w(x_j) - w(x_{j-1})| = \lim_{k \to +\infty} \left(\sum_{j=1}^{N} |w_{n_k}(x_j) - w_{n_k}(x_{j-1})| \right) \leqslant C,$$

and this concludes the proof. □

Theorem 2.5.4. *Consider a sequence of functions* $w_n : [0, +\infty[\times J \to \mathbb{R}^n$. *Assume that there exist positive constants* C, L *and* M *such that:*

1. $\operatorname{Tot.Var.} w_n(t, \cdot) \leqslant C$ *for every* $n \in \mathbb{N}$ *and* $t \geqslant 0$*;*
2. $|w_n(t, x)| \leqslant M$ *for every* $n \in \mathbb{N}$, $x \in J$ *and* $t \geqslant 0$*;*
3. $\int_J |w_n(t, x) - w_n(s, x)| \, dx \leqslant L \, |t - s|$ *for every* $n \in \mathbb{N}$ *and* $t, s \geqslant 0$.

Then there exist a function $w \in L^1_{loc}([0, +\infty \times J; \mathbb{R}^n)$ *and a subsequence* w_{n_k} *such that*

1. $w_{n_k} \to w$ *in* $L^1_{loc}([0, +\infty \times J; \mathbb{R}^n)$ *as* $k \to +\infty$*;*
2. $\int_J |w(t,x) - w(s,x)|\, dx \leqslant L\,|t-s|$ *for every* $t, s \geqslant 0$.

Moreover the values of w *can be uniquely determined by setting*

$$w(t,x) = \lim_{y \to x^+} w(t,y)$$

for every $t \geqslant 0$ *and* $x \in \operatorname{Int} J$*. In this case we have*

1. *Tot.Var.* $w(t,\cdot) \leqslant C$ *for every* $t \geqslant 0$*;*
2. $|w(t,x)| \leqslant M$ *for every* $t \geqslant 0$ *and* $x \in J$.

Proof. By Helly's theorem, it is possible to find a subsequence w_{n_k} such that, for every $t \geqslant 0$, $t \in \mathbb{Q}$, $w_{n_k}(t,\cdot) \to w(t,\cdot)$ pointwise and hence in $L^1_{loc}(\mathbb{R};\mathbb{R}^n)$. Thus we have

$$\int_J |w(t,x) - w(s,x)|\, dx \leqslant L\,|t-s|\,, \quad \text{Tot.Var.}\, w(t,\cdot) \leqslant C,$$

and

$$|w(t,x)| \leqslant M$$

for every $t, s \in \mathbb{Q} \cap [0, +\infty[$, $x \in J$. Fix now $t \geqslant 0$ and consider a sequence $t_m \to t$. Define

$$u(t,\cdot) = \lim_{m \to +\infty} w(t_m, \cdot).$$

By the previous estimates, this limit exists and is independent from the sequence t_m. We deduce

$$\int_J |u(t,x) - u(s,x)|\, dx \leqslant L\,|t-s|\,, \quad \text{Tot.Var.}\, u(t,\cdot) \leqslant C,$$

and

$$|u(t,x)| \leqslant M$$

for every $t, s \in [0, +\infty[$, $x \in J$, eventually by modifying the function u on a set of measure zero.

Consider now, for $\varepsilon > 0$ sufficiently small,

$$w_\varepsilon(t,x) = \frac{1}{\varepsilon} \int_x^{x+\varepsilon} u(t,s) ds$$

and

$$\tilde{w}(t,x) = \lim_{\varepsilon \to 0^+} w^\varepsilon(t,x).$$

This function satisfies

$$w(t,x) = \lim_{y \to x^+} w(t,y).$$

This concludes the proof. □

2.5.1 BV Functions in $\mathbb{R}^n$

In this section we describe briefly the L^1 theory for BV functions; see [44]. Let Ω be an open subset of $\mathbb{R}^n$ and consider $w : \Omega \to \mathbb{R}$. We denote with $\mathcal{B}(\Omega)$ the σ-algebra of Borel sets of Ω and with $\mathcal{B}_c(\Omega)$ the set

$$\{B \in \mathcal{B}(\Omega) : B \text{ compactly embedded in } \Omega\}\,. \tag{2.5.32}$$

Definition 2.5.5. *We say that $\mu : \mathcal{B}_c(\Omega) \to \mathbb{R}$ is a Radon measure if it is countable additive and $\mu(\emptyset) = 0$. We denote with $\mathfrak{M}(\Omega)$ the set of all Radon measures on Ω.*

The following theorem holds.

Theorem 2.5.6. *Fix a Radon measure $\mu \in \mathfrak{M}(\Omega)$. There exist two positive and unique Borel measures $\mu^+, \mu^- : \mathcal{B}(\Omega) \to [0, +\infty]$ such that*

$$\mu(E) = \mu^+(E) - \mu^-(E) \tag{2.5.33}$$

for every $E \in \mathcal{B}_c(\Omega)$.

Proof. (Sketch.) For every $E \in \mathcal{B}_c(\Omega)$ define the Borel signed measure

$$\mu \llcorner E : \mathcal{B}(\Omega) \to \mathbb{R},$$

by $(\mu \llcorner E)(B) = \mu(E \cap B)$ for every $B \in \mathcal{B}(\Omega)$. Thus we may consider the Jordan decomposition

$$\mu \llcorner E = (\mu \llcorner E)^+ - (\mu \llcorner E)^-.$$

If $E, F \in \mathcal{B}_c(\Omega)$ and $E \subseteq F$, then

$$(\mu \llcorner E)^+(B) = (\mu \llcorner F)^+(B) \quad \text{and} \quad (\mu \llcorner E)^-(B) = (\mu \llcorner F)^-(B)$$

for every $B \in \mathcal{B}(\Omega)$, $B \subseteq E$.

Take a sequence of open sets $(\Omega_k)_{k\in\mathbb{N}}$ such that $\overline{\Omega}_k \subseteq \Omega_{k+1}$ for every $k \in \mathbb{N}$ and $\Omega = \cup_{k\in\mathbb{N}}\Omega_k$. For every $B \in \mathcal{B}_c(\Omega)$ there exists $\bar{k} \in \mathbb{N}$ such that $B \subseteq \Omega_{\bar{k}}$. Hence we may define

$$\mu^+(B) = (\mu \llcorner \Omega_{\bar{k}})^+(B) \quad \text{and} \quad \mu^-(B) = (\mu \llcorner \Omega_{\bar{k}})^-(B).$$

This definition does not depend on the choice of $\Omega_{\bar{k}}$; therefore we have two positive measures μ^+ and μ^- defined on $\mathcal{B}_c(\Omega)$. It is also possible to extend in a unique way μ^+ and μ^- to $\mathcal{B}(\Omega)$ by taking

$$\mu^+(B) = \lim_{k\to+\infty} \mu^+(B \cap \Omega_k) \quad \text{and} \quad \mu^-(B) = \lim_{k\to+\infty} \mu^-(B \cap \Omega_k)$$

where $B \in \mathcal{B}(\Omega)$. Thus (2.5.33) holds. For the remaining part of the proof, we refer to [44] or to [45]. $\square$

We are now ready to give the definitions of bounded Radon measures and of bounded total variation functions.

Definition 2.5.7. *Fix a Radon measure $\mu \in \mathfrak{M}(\Omega)$ and consider the total variation of μ defined by $|\mu| := \mu^+ + \mu^-$. We say that μ has bounded total variation if $|\mu|(\Omega) < +\infty$ and we denote with $\mathfrak{M}_b(\Omega)$ the set of all Radon measures with bounded total variation.*

Remark 2.5.8. Notice that $\mathfrak{M}_b(\Omega)$ is a Banach space with respect to the norm

$$\|\mu\|_{\mathfrak{M}_b(\Omega)} = |\mu|(\Omega).$$

Definition 2.5.9. *We say that $w : \Omega \to \mathbb{R}$ has bounded total variation if*

1. *$w \in L^1(\Omega)$;*
2. *the i-th partial derivative $D_i w$ exists in the sense of distributions and belongs to $\mathfrak{M}_b(\Omega)$, for every $i = 1, \ldots, n$.*

The total variation of w is given by

$$\sum_{i=1}^{n} |D_i w|(\Omega).$$

We denote with $BV(\Omega)$ the set of all functions defined on Ω with bounded total variation.

Remark 2.5.10. The space $BV(\Omega)$ is a Banach space with respect to the norm

$$\|w\|_{L^1(\Omega)} + \sum_{i=1}^{n} |D_i w|(\Omega).$$

Remark 2.5.11. Let $w \in L^1(\Omega)$. Then $w \in BV(\Omega)$ if and only if there exists $c \in (0, +\infty)$ such that

$$\left| \int_{\Omega} w \operatorname{div} \varphi dx \right| \leqslant c \sup_{x \in \Omega} |\varphi(x)|$$

for every $\varphi \in C_c^{\infty}(\Omega; \mathbb{R}^n)$. In this case one can choose the constant c equal to the total variation of w.

Remark 2.5.12. If Ω is an interval of $\mathbb{R}$, then the two descriptions of BV functions are not completely equivalent. The most important difference is that if we change the values of a BV function w in a finite set, then the total variation of w changes but remain finite if we consider the first description, while it does not vary in the second case. Therefore, if we are interested only in the L^1 equivalence class, then we can assume that a BV function w is right continuous or left continuous.

2.6 Wave-Front Tracking and Existence of Solutions

This section deals with the existence of an entropy admissible solution to the Cauchy problem

$$\begin{cases} u_t + [f(u)]_x = 0, \\ u(0,\cdot) = \bar{u}(\cdot), \end{cases} \tag{2.6.34}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ is a smooth flux and $\bar{u} \in L^1(\mathbb{R}^n)$ is bounded in total variation. In order to prove existence, we construct a sequence of approximate solutions using the method called wave-front tracking algorithm.
We start considering the scalar case, while the system case, much more delicate, will be only sketched.

2.6.1 The Scalar Case

We assume the following conditions:

(C1) $f : \mathbb{R} \to \mathbb{R}$ is a scalar smooth function;
(C2) the characteristic field is either genuinely nonlinear or linearly degenerate.

It is possible to choose a sequence of piecewise constant functions $(\bar{u}_\nu)_\nu$ such that

$$\text{Tot.Var.}\{\bar{u}_\nu\} \leqslant \text{Tot.Var.}\{\bar{u}\}, \tag{2.6.35}$$

$$\|\bar{u}_\nu\|_{L^\infty} \leqslant \|\bar{u}\|_{L^\infty} \tag{2.6.36}$$

and

$$\|\bar{u}_\nu - \bar{u}\|_{L^1} < \frac{1}{\nu}, \tag{2.6.37}$$

for every $\nu \in \mathbb{N}$; see Figure 2.12.

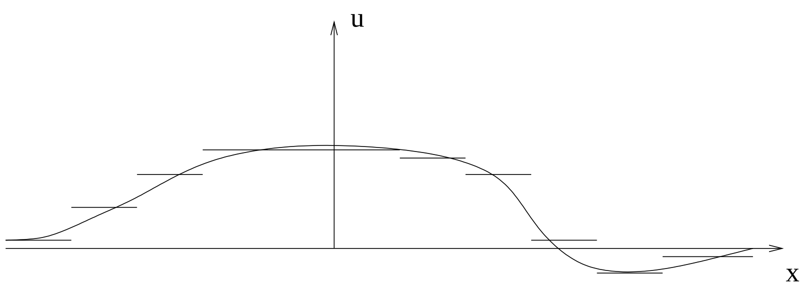


Fig. 2.12. A piecewise constant approximation of the initial datum satisfying (2.6.36) and (2.6.37).

Fix $\nu \in \mathbb{N}$. By (2.6.35), $\bar{u}_\nu$ has a finite number of discontinuities, say $x_1 < \cdots < x_N$. For each $i = 1, \dots, N$, we approximately solve the Riemann problem generated by the jump $(\bar{u}_\nu(x_i-), \bar{u}_\nu(x_i+))$ with piecewise constant functions of the type $\varphi(\frac{x-x_i}{t})$, where $\varphi : \mathbb{R} \to \mathbb{R}$. More precisely, if the Riemann problem generated by $(\bar{u}_\nu(x_i-), \bar{u}_\nu(x_i+))$ admits an exact solution containing just shocks or contact discontinuities, then $\varphi(\frac{x-x_i}{t})$ is the exact solution, while if a rarefaction wave appears, then we split it in a centered rarefaction fan, containing a sequence of jumps of size at most $\frac{1}{\nu}$, travelling with a speed between the characteristic speeds of the states connected. In this way, we are able to construct an approximate solution $u_\nu(t, x)$ until a time t_1, where at least two wave fronts interact together; see Figure 2.13.

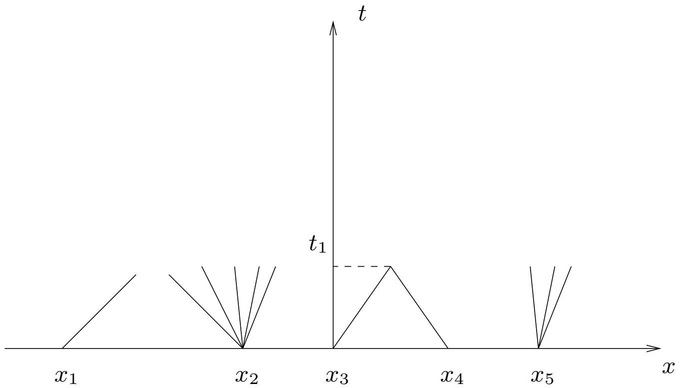


Fig. 2.13. The wave front tracking construction until the first time of interaction.

Remark 2.6.1. In the scalar case, if the characteristic field is linearly degenerate, then all the waves are contact discontinuities and travel at the same speed. Therefore the previous construction can be done for every positive time.

Remark 2.6.2. Notice that it is possible to avoid that three of more wave fronts interact together at the same time slightly changing the speed of some wave fronts. This may introduce a small error of the approximate solution with respect to the exact one.

At time $t = t_1$, $u_\nu(t_1, \cdot)$ is clearly a piecewise constant function. So we can repeat the previous construction until a second interaction time $t = t_2$ and so on. In order to prove that a wave-front tracking approximate solution exists for every $t \in [0, T]$, where T may be also $+\infty$, we need to estimate

1. the number of waves;
2. the number of interactions between waves;
3. the total variation of the approximate solution.

The first two estimates are concerned with the possibility to construct a piecewise constant approximate solution. The third estimate, instead, is concerned with the convergence of the approximate solutions towards an exact solution.

Remark 2.6.3. The two first bounds are nontrivial for the vector case and it is necessary to introduce simplified solutions to Riemann problems and/or non-physical waves.

The next lemma shows that the number of interactions is finite.

Lemma 2.6.4. *The number of wave fronts for the approximate solution u_ν is not increasing with respect to the time and so u_ν is defined for every $t \geqslant 0$. Moreover the number of interactions between waves is bounded by the number of wave fronts.*

Proof. Consider two wave fronts interacting together. The wave fronts can be:

1. two shocks,
2. two rarefaction shocks,
3. a shock and a rarefaction shock,
4. two contact discontinuities.

By Remark 2.6.1, the case of two contact discontinuities can not happen. Moreover, the speeds of waves imply that also the case of two rarefaction shocks can not happen. In fact, suppose that two rarefaction shocks interact together at a certain time. Denote with u_l, u_m and u_r respectively the states as in Figure 2.14. Since these waves are rarefactions shocks, we have

$$\lambda(u_l) < \lambda(u_m) < \lambda(u_r),$$

where λ denotes the characteristic speeds of the states. Therefore the wave connecting u_l to u_m has a speed less than or equal to the speed of the wave connecting u_m to u_r and the wave fronts can not interact together.

So the remaining possibilities are the followings.

1. Two shocks. In this case it is clear that after the interaction, a single shock wave is created. So the number of waves decreases by 1.
2. A shock and a rarefaction shock. In this case either a single shock wave is produced as in the previous possibility, or a single rarefaction shock is created. In fact, if the exact solution to the Riemann problem at the interaction time is given by a rarefaction wave, then the size of the rarefaction wave is less than or equal to the size of the rarefaction shock, which is less than or equal to $1/\nu$. This implies that the wave is split in a single rarefaction shock. Thus the number of waves decreases by 1.

Therefore we conclude that a each interaction the number of wave fronts decreases at least by 1 and so the lemma is proved. □

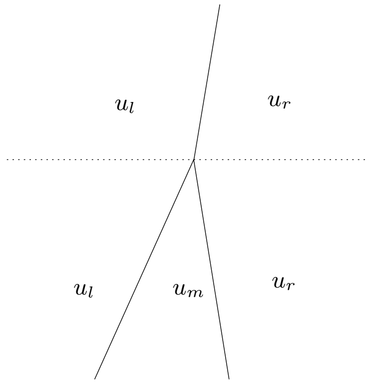


Fig. 2.14. Interaction between two wave fronts.

Lemma 2.6.5. *The total variation of $u_\nu(t,\cdot)$ is not increasing with respect the time. Therefore for each $t \geqslant 0$*

$$Tot.Var.\, u_\nu(t,\cdot) \leqslant Tot.Var.\, \bar{u}. \tag{2.6.38}$$

Proof. It is clear that the total variation may vary only at interaction times.

Consider an interaction of two wave fronts at time $\bar{t}$. Let us call by u_l, u_m and u_r respectively the left, the middle and the right state of the wave fronts; see Figure 2.14.

The interaction between the two waves produces a single wave connecting u_l with u_r. The variation before $t = \bar{t}$ due to the interacting waves is given by $|u_l - u_m| + |u_m - u_r|$, while the variation after $t = \bar{t}$ due to the wave produced is given by $|u_l - u_r|$. Triangular inequality implies that

$$|u_l - u_r| \leqslant |u_l - u_m| + |u_m - u_r|$$

and so the proof is finished. □

The following theorem holds.

Theorem 2.6.6. *Let $f : \mathbb{R} \to \mathbb{R}$ be smooth and $\bar{u} \in L^1(\mathbb{R})$ with bounded variation. Then there exists an entropy-admissible solution $u(t,x)$ to the Cauchy problem (2.6.34), defined for every $t \geqslant 0$. Moreover*

$$\|u(t,\cdot)\|_{L^\infty} \leqslant \|\bar{u}(\cdot)\|_{L^\infty} \tag{2.6.39}$$

for every $t \geqslant 0$.

Proof. For every $\nu \in \mathbb{N}$, construct a wave-front tracking approximate solution u_ν as before in this section.

Clearly we have

$$|u_\nu(t,x)| \leqslant |u_\nu(0,x)| \leqslant \|\bar{u}\|_{L^\infty} \tag{2.6.40}$$

for every $\nu \in \mathbb{N}$, $t \geqslant 0$ and $x \in \mathbb{R}$. By Lemma 2.6.5,

$$\text{Tot.Var.}\, u_\nu(t,\cdot) \leqslant \text{Tot.Var.}\, \bar{u}, \tag{2.6.41}$$

for every $t \geqslant 0$ and $\nu \in \mathbb{N}$. Finally the maps $t \mapsto u_\nu(t,\cdot)$ are uniformly Lipschitz continuous with values in $L^1(\mathbb{R};\mathbb{R})$. Therefore, by Theorem 2.5.4, we can extract a subsequence, denoted again by $u_\nu(t,x)$, converging to some function $u(t,x)$ in $L^1([0,+\infty[\times\mathbb{R};\mathbb{R})$. Since $\|u_\nu(0,\cdot) - \bar{u}(\cdot)\|_{L^1} \to 0$, then the initial condition clearly holds.

It remains to prove that $u(t,x)$ is a weak solution to the Cauchy problem (2.6.34) and that it is entropy admissible. To prove the first claim, fix $T > 0$ and an arbitrary C^1 function ψ with compact support in $]-\infty, T[\times\mathbb{R}$. We need to prove that

$$\int_0^T \int_{\mathbb{R}} \{u \cdot \psi_t + f(u) \cdot \psi_x\} dxdt + \int_{\mathbb{R}} \bar{u}(x) \cdot \psi(0,x) dx = 0.$$

It is sufficient to prove that

$$\lim_{\nu \to +\infty} \left\{ \int_0^T \int_{\mathbb{R}} \{u_\nu \cdot \psi_t + f(u_\nu) \cdot \psi_x\} dxdt + \int_{\mathbb{R}} u_\nu(0,x) \cdot \psi(0,x) dx \right\} = 0. \tag{2.6.42}$$

Fix $\nu \in \mathbb{N}$. At every $t \in [0,T]$, call $x_1(t) < \cdots < x_N(t)$ the points where $u_\nu(t,\cdot)$ has a jump and set

$$\begin{aligned} \Delta u_\nu(t,x_\alpha) &:= u_\nu(t,x_\alpha+) - u_\nu(t,x_\alpha-), \\ \Delta f(u_\nu(t,x_\alpha)) &:= f(u_\nu(t,x_\alpha+)) - f(u_\nu(t,x_\alpha-)). \end{aligned}$$

The lines $x_\alpha(t)$ divide $[0,T] \times \mathbb{R}$ into a finite number of regions, say Γ_j, where u_ν is constant. Applying the divergence theorem to $(\psi \cdot u_\nu, \psi \cdot f(u_\nu))$ and splitting the integral (2.6.42) over the regions Γ_j, we obtain that the integral (2.6.42) can be rewritten in the form

$$\int_0^T \sum_\alpha \left[\dot{x}_\alpha(t) \cdot \Delta u_\nu(t,x_\alpha) - \Delta f(u_\nu(t,x_\alpha))\right] \psi(t,x_\alpha(t)) dt. \tag{2.6.43}$$

If x_α is a shock wave or a contact discontinuity, then

$$\dot{x}_\alpha(t) \cdot \Delta u_\nu(t,x_\alpha) - \Delta f(u_\nu(t,x_\alpha)) = 0,$$

while if x_α is a rarefaction wave, then

$$\dot{x}_\alpha(t) \cdot \Delta u_\nu(t,x_\alpha) - \Delta f(u_\nu(t,x_\alpha)),$$

depends linearly on the L^∞ distance between u_ν and $\bar{u}$. Splitting the summation in (2.6.43) over waves of the same type, we deduce that the previous integral tends to 0 as $\nu \to +\infty$, concluding that $u(t,x)$ is a weak solution to the Cauchy problem.

Fix now η a convex entropy with a corresponding entropy flux q. It remains to prove that

$$\liminf_{\nu\to+\infty} \int_0^T \int_{\mathbb{R}} [\eta(u_\nu)\psi_t + q(u_\nu)\psi_x]\, dxdt \geqslant 0$$

for every C^1 positive function φ with compact support. Using again the divergence theorem as before, we need to prove that

$$\liminf_{\nu\to+\infty} \int_0^T \sum_\alpha [\dot{x}_\alpha(t) \cdot \Delta\eta(u_\nu(t,x_\alpha)) - \Delta q(u_\nu(t,x_\alpha))]\, \varphi(t,x_\alpha)dt \geqslant 0$$

where

$$\begin{aligned}\Delta\eta(u_\nu(t,x_\alpha)) &:= \eta(u_\nu(t,x_\alpha+)) - \eta(u_\nu(t,x_\alpha-)),\\ \Delta q(u_\nu(t,x_\alpha)) &:= q(u_\nu(t,x_\alpha+)) - q(u_\nu(t,x_\alpha-)).\end{aligned}$$

Using the same estimates as in the previous case, we conclude. □

2.6.2 The System Case

For systems, the construction of wave-front tracking approximations is more complex, because more types of interactions may happen. In particular the bounds on number of waves, interactions and BV norms are no more directly obtained.

Let us start giving some total variation estimates for interaction of waves along a wave-front tracking approximation. These permit to illustrate the ideas for obtaining the needed bounds in system case. The constants in the estimates depend on the total variation of the initial data, which is assumed to be sufficiently small.

Consider a wave of the i–th family of strength σ_i interacting with a wave of the j–th family of strength σ_j, $i \neq j$, and indicate by σ'_k ($k \in \{1,\dots,n\}$) the strengths of the new waves produced by the interaction. Then it holds

$$|\sigma_i - \sigma'_i| + |\sigma_j - \sigma'_j| + \sum_{k\neq i,j} |\sigma'_k| \leq C|\sigma_i||\sigma_j|, \tag{2.6.44}$$

For the case $i = j$, let us indicate by $\sigma_{i,1}$ and $\sigma_{i,2}$ the strengths of the interacting waves, then it holds

$$|\sigma_{i,1} + \sigma_{i,2} - \sigma'_i| + \sum_{k\neq i} |\sigma'_k| \leq C|\sigma_{i,1}||\sigma_{i,2}|. \tag{2.6.45}$$

One can now fix a parameter δ_ν and split rarefactions in rarefaction fans with shocks of strength at most δ_ν. Also, at each interaction time, one solves exactly the new Riemann problem, eventually splitting the rarefaction waves in rarefaction fans, only if the product of interacting waves is bigger than δ_ν. Otherwise, one solves the Riemann problem only with waves of the same families of the interacting ones, the error being transported along a *non-physical wave*, travelling at a speed bigger than all waves. In this way, it is possible to control the number of waves and interactions and then let δ_ν go to zero. For details see [19].

Consider now a wave-front tracking approximate solution u_ν and let $x_\alpha(t)$, of family i_α and strength σ_α, indicate the discontinuities of $u_\nu(t)$. We say that two discontinuities are interacting if $x_\alpha < x_\beta$ and either $i_\alpha > i_\beta$ or $i_\alpha = i_\beta$ and at least one of the two waves is a shock. We define the Glimm functional computed at $u_\nu(t)$ as:

$$Y(u_\nu(t)) = TV(u_\nu(t)) + C_1\, Q(u_\nu(t)),$$

where C_1 is a constant to be chosen suitably and

$$Q(u_\nu(t)) = \sum |\sigma_\alpha||\sigma_\beta|$$

where the sum is over interacting waves. One can easily prove that the functional Y is equivalent to the functional measuring the total variation. Clearly such functional changes only at interaction times. Using the interaction estimates (2.6.44) and (2.6.45), at an interaction time $\bar{t}$, we get

$$|TV(u_\nu(\bar{t}+)) - TV(u_\nu(\bar{t}-))| \leqslant C|\sigma_i||\sigma_j|,$$

$$Q(u_\nu(\bar{t}+)) - Q(u_\nu(\bar{t}-)) \leqslant -C_1|\sigma_i||\sigma_j| + C|\sigma_i||\sigma_j|\; TV(u_\nu(\bar{t}-)).$$

Therefore

$$Y(u_\nu(\bar{t}+)) - Y(u_\nu(\bar{t}-)) \leqslant |\sigma_i||\sigma_j|\big[C - C_1 + C\; TV(u_\nu(\bar{t}-))\big].$$

On the other side, for every t:

$$TV(u_\nu(t)) \leq Y(u_\nu(t)).$$

Then choosing $C_1 > C$ and assuming that $TV(u_\nu(0))$ is sufficiently small, one has that Y is decreasing along a wave-front tracking approximate solution and so the total variation is controlled.

2.7 Uniqueness and Continuous Dependence

The aim of this section is to illustrate a method to prove uniqueness and Lipschitz continuous dependence by initial data for solutions to the Cauchy problem, controlling for any two approximate solutions u, u' how their distance varies in time. For simplicity we restrict to the scalar case. The method was introduced in [18] and is based on a Riemannian type distance on L^1. In [20], the approach was applied to systems case. Various alternative methods were recently introduced to treat uniqueness, see the bibliographical note, but the one presented here is the more suitable to be used for networks.

The basic idea is to estimate the L^1-distance viewing L^1 as a Riemannian manifold. We consider the subspace of piecewise constant functions and "generalized tangent vectors" consisting of two components (v, ξ), where $v \in L^1$ describes the L^1 infinitesimal displacement, while $\xi \in \mathbb{R}^n$ describes the infinitesimal displacement of discontinuities. For example, take a family of piecewise constant functions $\theta \to u^\theta$, $\theta \in [0,1]$, each of which has the same number of jumps, say at the points $x_1^\theta < ... < x_N^\theta$. Assume that the following functions are well defined (see Figure 2.15)

$$L^1 \ni v^\theta(x) \doteq \lim_{h\to 0} \frac{u^{\theta+h}(x) - u^\theta(x)}{h},$$

and also the numbers

$$\xi_\beta^\theta \doteq \lim_{h\to 0} \frac{x_\beta^{\theta+h} - x_\beta^\theta}{h}, \qquad \beta = 1, ..., N.$$

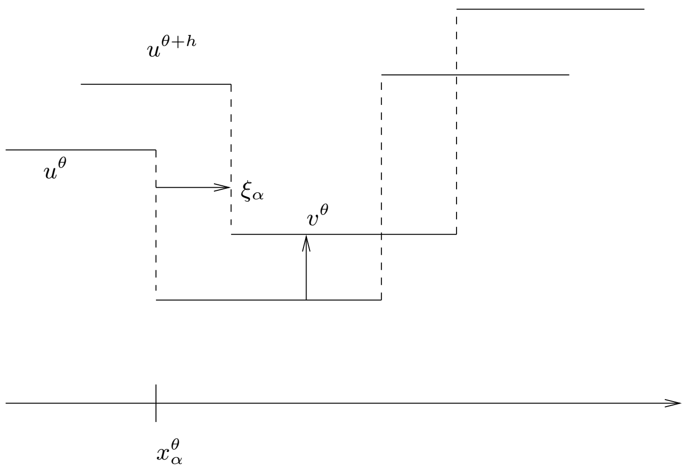


Fig. 2.15. Construction of "generalized tangent vectors".

Then we say that γ admits tangent vectors $(v^\theta, \xi^\theta) \in T_{u^\theta} \doteq L^1(\mathbb{R};\mathbb{R}^n) \times \mathbb{R}^n$. In general such path $\theta \to u^\theta$ is not differentiable w.r.t. the usual differential structure of L^1, in fact if $\xi_\beta^\theta \neq 0$, as $h \to 0$ the ratio $\left[u^{\theta+h}(x) - u^\theta(x)\right]/h$ does not converge to any limit in L^1.
One can compute the L^1-length of the path $\gamma : \theta \to u^\theta$ in the following way:

$$\|\gamma\|_{L^1} = \int_0^1 \left\|v^\theta\right\|_{L^1} d\theta + \sum_{\beta=1}^{N} \int_0^1 \left|u^\theta(x_\beta+) - u^\theta(x_\beta-)\right| \left|\xi_\beta^\theta\right| d\theta. \tag{2.7.46}$$

According to (2.7.46), in order to compute the L^1-length of a path γ, we integrate the norm of its tangent vector which is defined as follows:

$$\|(v,\xi)\| \doteq \|v\|_{L^1} + \sum_{\beta=1}^{N} |\Delta u_\beta| \, |\xi_\beta| \, ,$$

where $\Delta u_\beta = u(x_\beta+) - u(x_\beta-)$ is the jump across the discontinuity x_β.

Let us introduce the following definition.

Definition 2.7.1. *We say that a continuous map $\gamma : \theta \to u^\theta \doteq \gamma(\theta)$ from $[0,1]$ into L^1_{loc} is a regular path if the following holds. All functions u^θ are piecewise constant, with the same number of jumps, say at $x_1^\theta < ... < x_N^\theta$ and coincide outside some fixed interval $]-M, M[$. Moreover, γ admits a generalized tangent vector $D\gamma(\theta) = (v^\theta, \xi^\theta) \in T_{\gamma(\theta)} = L^1(\mathbb{R};\mathbb{R}^n) \times \mathbb{R}^N$, continuously depending on θ .*

Given two piecewise constant functions u and u', call $\Omega(u,u')$ the family of all regular paths $\gamma : [0,1] \to \gamma(t)$ with $\gamma(0) = u$, $\gamma(1) = u'$. The Riemannian distance between u and u' is given by

$$d(u,u') \doteq \inf\left\{\|\gamma\|_{L^1}, \gamma \in \Omega(u,u')\right\}.$$

To define d on all L^1, for given $u, u' \in L^1$ we set

$$\begin{aligned} d(u,u') \doteq \inf\{&\|\gamma\|_{L^1} + \|u - \tilde{u}\|_{L^1} + \|u' - \tilde{u}'\|_{L^1} : \\ &\tilde{u}, \tilde{u}' \text{ piecewise constant functions}, \gamma \in \Omega(u,u')\}. \end{aligned}$$

It is easy to check that this distance coincides with the distance of L^1. (For the systems case, one has to introduce weights, see [20], obtaining an equivalent distance.)

Now we are ready to estimate the L^1 distance among solutions, studying the evolution of norms of tangent vectors along wave-front tracking approximations. Take u, u' piecewise constant functions and let $\gamma_0(\vartheta) = u^\vartheta$ be a regular path joining $u = u^0$ with $u' = u^1$. Define $u^\vartheta(t,x)$ to be a wave-front tracking approximate solution with initial data u^ϑ and let $\gamma_t(\vartheta) = u^\vartheta(t,\cdot)$.

One can easily check that, for every γ_0 (regular path) and every $t \geq 0$, γ_t is a regular path. If we can prove

$$\|\gamma_t\|_{L^1} \leq \|\gamma_0\|_{L^1}, \tag{2.7.47}$$

then for every $t \geq 0$

$$\|u(t,\cdot) - u'(t,\cdot)\|_{L^1} \leq \inf_{\gamma_t} \|\gamma_t\|_{L^1} \leq \inf_{\gamma_0} \|\gamma_0\|_{L^1} = \|u(0,\cdot) - u'(0,\cdot)\|_{L^1}. \tag{2.7.48}$$

To obtain (2.7.47), hence (2.7.48), it is enough to prove that, for every tangent vector $(v, \xi)(t)$ to any regular path γ_t, one has:

$$\|(v,\xi)(t)\| \leq \|(v,\xi)(0)\|\,, \tag{2.7.49}$$

i.e the norm of a tangent vector does not increase in time. Moreover, if (2.7.48) is established, then uniqueness and Lipschitz continuous dependence of solutions to Cauchy problems is straightforwardly achieved passing to the limit on the wave-front tracking approximate solutions.

Let us now estimate the increase of the norm of a tangent vector. In order to achieve (2.7.49), we fix a time $\bar{t}$ and treat the following cases:

Case 1. no interaction of waves takes place at $\bar{t}$;
Case 2. two waves interact at $\bar{t}$;

In Case 1, denote by x_β, σ_β, and ξ_β, respectively, the positions, sizes and shifts of the discontinuities of a wave-front tracking approximate solution. Following [20] we get:

$$\frac{d}{dt}\left\{\int |v(t,x)|\,dx + \sum_\beta |\xi_\beta|\,|\sigma_\beta|\right\} =$$
$$-\left\{\sum_\beta \left(\lambda(\rho^-) - \dot{x}_\beta\ \right)\left|v^-\right| + \sum_\beta \left(\dot{x}_\beta - \lambda(\rho^+)\right)\left|v^+\right|\right\} +$$
$$+\sum_\beta D\lambda(\rho^-,\rho^+)\cdot(v^-,v^+)\,(\mathrm{sign}\xi_\beta)\,|\sigma_\beta|\,,$$

with $\sigma_\beta = \rho^+ - \rho^-$, $\rho^\pm \doteq \rho(x_\beta\pm)$ and similarly for $v^\pm$. If the waves respect the Rankine-Hugoniot conditions, then

$$D\lambda(\rho^-,\rho^+)(v^-,v^+) = \left(\lambda(\rho^-) - \dot{x}_\beta\right)\frac{v^-}{|\sigma_\beta|} + \left(\dot{x}_\beta - \lambda(\rho^+)\right)\frac{v^+}{|\sigma_\beta|}$$

and

$$\frac{d}{dt}\left\{\int |v(t,x)|\,dx + \sum_\beta |\xi_\beta|\,|\sigma_\beta|\right\} \leq 0. \tag{2.7.50}$$

In the wave front tracking algorithm the Rankine-Hugoniot condition may be violated for rarefaction fans. However, this results in an increase of the distance which is controlled in terms of $1/\nu$ (the size of a rarefaction shock) and tends to zero with $\nu \to \infty$.

Let us now pass to Case 2. First, we have the following:

Lemma 2.7.2. *Consider two waves, with speeds λ_1 and λ_2 respectively, that interact together at $\bar{t}$ producing a wave with speed λ_3. If the first wave is shifted* *by* ξ_1 *and the second wave by* ξ_2*, then the shift of the resulting wave is given by*

$$\xi_3 = \frac{\lambda_3 - \lambda_2}{\lambda_1 - \lambda_2}\xi_1 + \frac{\lambda_1 - \lambda_3}{\lambda_1 - \lambda_2}\xi_2. \tag{2.7.51}$$

Moreover we have that

$$\Delta\rho_3\,\xi_3 = \Delta\rho_1\,\xi_1 + \Delta\rho_2\,\xi_2, \tag{2.7.52}$$

where $\Delta\rho_i$ *are the signed strengths of the corresponding waves.*

Proof. We suppose that ρ_l and ρ_m are the left and the right values of the wave with speed λ_1 and ρ_m and ρ_r are the left and the right values of the wave with speed λ_2, see Figure 2.16. So $\Delta\rho_1 = \rho_m - \rho_l$, $\Delta\rho_2 = \rho_r - \rho_m$ and

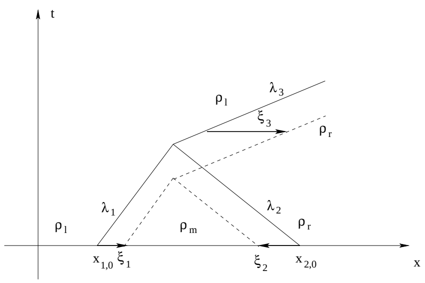


Fig. 2.16. Shifts of waves.

$\Delta\rho_3 = \rho_r - \rho_l$. The two wave fronts have respectively equation

$$x = \lambda_1 t + x_{1,0}, \qquad x = \lambda_2 t + x_{2,0},$$

where $x_{1,0}$ and $x_{2,0}$ are the initial positions of the wave fronts with speed λ_1 and λ_2 respectively. Therefore they interact at the point

$$(\bar{x}, \bar{t}) = \left(x_{1,0} + \lambda_1 \frac{x_{1,0} - x_{2,0}}{\lambda_2 - \lambda_1}, \frac{x_{1,0} - x_{2,0}}{\lambda_2 - \lambda_1} \right).$$

If we consider the shifts, then the two wave fronts interact at the point

$$(\tilde{x}, \tilde{t}) = \left(x_{1,0} + \xi_1 + \lambda_1 \frac{(x_{2,0} + \xi_2) - (x_{1,0} + \xi_1)}{\lambda_1 - \lambda_2}, \frac{(x_{2,0} + \xi_2) - (x_{1,0} + \xi_1)}{\lambda_1 - \lambda_2} \right).$$

The shift of the new wave is thus:

$$\xi_3 = \tilde{x} + \lambda_3(\bar{t} - \tilde{t}) - \bar{x} = \xi_1 + \lambda_1 \frac{\xi_2 - \xi_1}{\lambda_1 - \lambda_2} + \lambda_3 \frac{\xi_1 - \xi_2}{\lambda_1 - \lambda_2}$$

and consequently (2.7.51) holds. Multiplying equation (2.7.51) by $\Delta\rho_3 = \Delta\rho_1 + \Delta\rho_2$, using $\Delta f_3 = \Delta f_1 + \Delta f_2$ (the jumps in the fluxes) and the Rankine-Hugoniot condition $\lambda = \Delta f / \Delta\rho$, we get:

$$\Delta\rho_3\xi_3 = \Delta\rho_3 \frac{\frac{\Delta f_3}{\Delta\rho_3} - \frac{\Delta f_2}{\Delta\rho_2}}{\frac{\Delta f_1}{\Delta\rho_1} - \frac{\Delta f_2}{\Delta\rho_2}}\xi_1 + \Delta\rho_3 \frac{\frac{\Delta f_1}{\Delta\rho_1} - \frac{\Delta f_3}{\Delta\rho_3}}{\frac{\Delta f_1}{\Delta\rho_1} - \frac{\Delta f_2}{\Delta\rho_2}}\xi_2 =$$

$$= \frac{\Delta f_3 \Delta\rho_2 - \Delta f_2 \Delta\rho_3}{\Delta f_1 \Delta\rho_2 - \Delta f_2 \Delta\rho_1}\Delta\rho_1\xi_1 + \frac{\Delta f_1 \Delta\rho_3 - \Delta f_3 \Delta\rho_1}{\Delta f_1 \Delta\rho_2 - \Delta f_2 \Delta\rho_1}\Delta\rho_2\xi_2 =$$

$$= \Delta\rho_1\xi_1 + \Delta\rho_2\xi_2.$$

This concludes the proof. □

From (2.7.52) it follows

$$|\Delta\rho_3\xi_3| \leq |\Delta\rho_1|\,|\xi_1| + |\Delta\rho_2|\,|\xi_2|\,,$$

from which

$$\|(v,\xi)(\bar{t}+)\| \leq \|(v,\xi)(\bar{t}-)\|\,. \tag{2.7.53}$$

2.8 Exercises

Exercise 2.8.1. Consider the Burgers equation

$$u_t + uu_x = 0, \tag{2.8.54}$$

with the initial condition

$$u(0,x) = u_0(x), \tag{2.8.55}$$

where $u_0 \in C^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$.

1. Prove that the characteristic curve for (2.8.54) are lines whose slope is given by u_0 evaluated at the intersection point of the characteristic with

$$\{(x,t) \in \mathbb{R} \times [0,+\infty[: t = 0\}\,.$$

2. Suppose that $\inf u_0'(x) \geqslant 0$. Prove that there exists a C^1 solution $u(t,x)$ to (2.8.54) and (2.8.55) for every $t \geqslant 0$.
3. Define $L := \inf u_0'(x)$. Assume $L < 0$. Prove that for every $0 < \bar{t} < -\frac{1}{L}$ there exists a C^1 solution $u(t,x)$ to (2.8.54) and (2.8.55) for $t \in [0,\bar{t}]$.

Exercise 2.8.2. Assume that $f : \mathbb{R} \to \mathbb{R}$ is a smooth function. For every $\varepsilon > 0$ consider a function u^ε solution to

$$(u^\varepsilon)_t + f(u^\varepsilon)_x = \varepsilon(u^\varepsilon)_{xx}. \tag{2.8.56}$$

Suppose that u^ε converges in $L^1_{loc}(\mathbb{R})$ as $\varepsilon \to 0$ to a function u.

1. Prove that u is a weak solution to

$$u_t + f(u)_x = 0.$$

2. Prove that u is entropy admissible.
   Hint: multiply both sides of (2.8.56) by $\eta'(u^\varepsilon)$, where η is a convex entropy, integrate and pass to the limit as $\varepsilon \to 0$; see also [19].

Exercise 2.8.3. Consider the functions

$$\eta(u) = |u - k|, \quad q(u) = \text{sgn}\,(u-k)\,(f(u) - f(k)), \tag{2.8.57}$$

where $k \in \mathbb{R}$, $u \in \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ smooth function. Prove that η is a C^0 entropy and q is the corresponding flux.

Exercise 2.8.4. Consider the conservation law

$$u_t + f(u)_x = 0,$$

where $f : \mathbb{R} \to \mathbb{R}$ is a smooth function. Prove that:

1. the characteristic field is linearly degenerate if and only if f is an affine function;
2. the characteristic field is genuinely nonlinear if and only if f is a convex or concave function.

Exercise 2.8.5. Consider the scalar Cauchy problem

$$\begin{cases} u_t + f(u)_x = 0, \\ u(0,x) = u_0(x), \end{cases}$$

where the flux $f : \mathbb{R} \to \mathbb{R}$ is a convex (resp. concave) C^1 function. Prove that a piecewise C^1 solution u satisfying the Kruzkov entropy admissible condition also satisfies $u^- > u^+$ (resp. $u^- < u^+$) along every line of jump.

Moreover prove that (2.3.19) can be rewritten in the form

$$\frac{f(u^+) - f(u^*)}{u^+ - u^*} \leqslant \frac{f(u^-) - f(u^*)}{u^- - u^*} \tag{2.8.58}$$

for every $u^* = \alpha u^+ + (1-\alpha)u^-$, $0 < \alpha < 1$.

2.9 Bibliographical Note

The theory of systems of conservation laws is extensively described in the books by Bressan [19], Dafermos [40], Holden and Risebro [67] and Serre [98].

The general solution to the Riemann problem for a strictly hyperbolic system of conservation laws was first obtained by Lax in [83]. A complete analysis of the Riemann problem for the p-system can be found in the book by Smoller [99].

For a discussion of entropy-admissible conditions, see Dafermos [40], Lax [83] and Smoller [99].

The first proof of global existence for weak entropic solution appeared in the seminal paper by Glimm [49]. It is based on a construction of approximate solutions generated by Riemann problems with a randomly restarting procedure.

For the scalar case, there is another proof, based on piecewise constant approximations, for the existence of an entropy admissible solution. This method is due to Dafermos; see [40].

The wave-front tracking method was first introduced by Di Perna in [42] and then it was extended by Bressan [19] and Risebro [96].

Uniqueness and Lipschitz continuous dependence of solutions to scalar conservation laws (in many space variables) was first obtained by Kruzkov using the special entropies (2.8.57), see [80]. Viscous approximations can also be used for the scalar case, see [98]. The first proof of uniqueness for systems was obtained in 1996 by Bressan et al., then published in [20]. The proof was much simplified using the Bressan-Liu-Yang functionals, see [21]. Finally, recently the viscous approximation was used also for systems in the spectacular results of Bianchini and Bressan, see [17].

3

Macroscopic Traffic Models

Macroscopic traffic models describe the evolution of vehicle positions in term of macroscopic variables as the density and the average speed of cars. An unidirectional road is modeled by an interval $I = [a, b]$ of $\mathbb{R}$ $(a < b)$ and so the density $\rho(t, x)$ and the average velocity $v(t, x)$ depend on the time t and on $x \in I$.

The simplest model is the scalar one proposed independently by Lighthill and Whitham in 1955 and by Richards in 1956. It is based on the conservation of cars and is described by a single equation in conservation form.

Then some second order models, i.e. with two equations, were proposed by Payne in 1971 and Whitham in 1974. Unfortunately Daganzo showed that all these second order models are not good to describe traffic flow. In particular he proved that cars may exhibit negative speed. Finally Aw and Rascle in 2000, to overcome Daganzo's observations, proposed a second order model, which became a starting point for a lot of other traffic models and derivations.

In this chapter we describe all these models together with more advanced ones proposed recently.

3.1 Lighthill-Whitham-Richards Model

The model is based on conservation of cars and it is described by the equation

$$\rho_t + f(\rho)_x = 0, \tag{3.1.1}$$

where the flux $f(\rho, v)$ is given by ρv. The average speed v is assumed to be a function depending only on the density. For simplicity, we suppose that

(LWR1) f is a C^2 function;
(LWR2) f is a strictly concave function;
(LWR3) $f(0) = f(\rho_{max}) = 0$.

The case of not strictly concave flux, of interest in real traffic applications, is illustrated below in Section 3.1.4.

Without loss of generality, assume that $\rho_{max} = 1$.

Example 3.1.1. If $\rho \in [0,1]$ and $v(\rho) = 1 - \rho$, then $f(\rho) = \rho(1-\rho)$. This flux function satisfies all the previous assumptions. In fact $f(0) = 0$, $f(1) = 0$ and $f''(\rho) = -2$.

3.1.1 Derivation of the Equation

The derivation of (3.1.1) is the following. Let $\rho(t,x)$ be the density of cars on an unidimensional road at the point $x \in \mathbb{R}$ and at time $t \geqslant 0$. Fix two arbitrary points $a < b$ on the road. The variation of the number of cars is due only to the interior flux at $x = a$ and to the exterior flux at $x = b$. Therefore

$$\begin{aligned}\frac{d}{dt}\int_a^b \rho(t,x)dx &= v(\rho(t,a))\rho(t,a) - v(\rho(t,b))\rho(t,b)\\ &= -\int_a^b \left[f(\rho(t,x))\right]_x dx. \end{aligned} \tag{3.1.2}$$

Since (3.1.2) holds for every $a < b$, then we deduce the scalar conservation law

$$\rho_t + f(\rho)_x = 0.$$

3.1.2 Fundamental Diagrams

The main assumption for the Lighthill-Whitham-Richards model is that the average velocity v depends only on the density of the cars. A reasonable property of v is that v is a decreasing function of the density. The law giving the flux as function of the density is called fundamental diagram.

We describe various fundamental diagrams assigning the velocity function $v = v(\rho)$, thus the flux is simply obtained multiplying by the density ρ. The simplest fundamental diagram is obtained setting v to be a linear function of the density, i.e.

$$v(\rho) = v_{max}\left(1 - \frac{\rho}{\rho_{max}}\right); \tag{3.1.3}$$

see Figure 3.1.

Another fundamental diagram was considered by Greenberg and was supported by experimental data from the Lincoln tunnel in New York. He proposed the velocity function

$$v(\rho) = v_0 \log\left(\frac{\rho_{max}}{\rho}\right), \tag{3.1.4}$$

where v_0 is a positive constant. In this case $v(\rho_{max}) = 0$, while v is unbounded when $\rho \to 0^+$; see Figure 3.2.

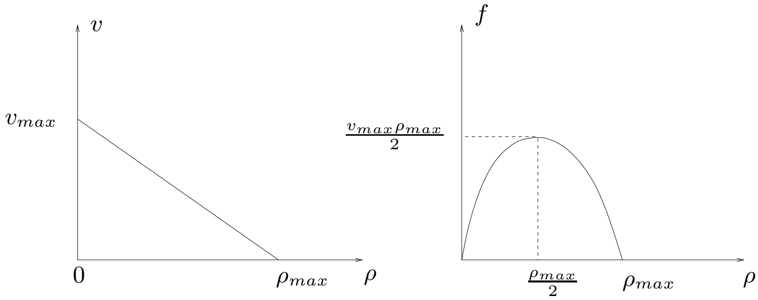


Fig. 3.1. The velocity function and the fundamental diagram for (3.1.3).

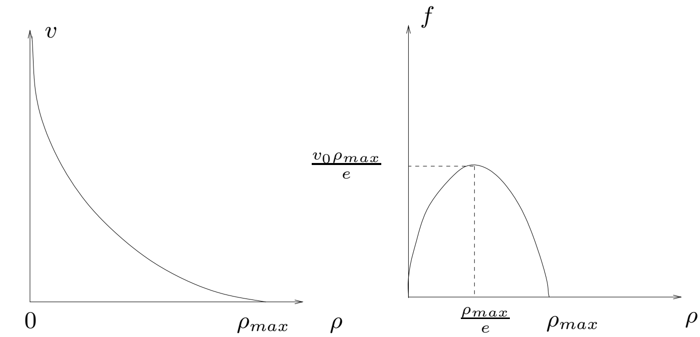


Fig. 3.2. The velocity function and the fundamental diagram when v is given by (3.1.4).

A third fundamental diagram is given by the Underwood model, whose velocity function is

$$v(\rho) = v_{max} e^{\left(-\frac{\rho}{\rho_{max}}\right)}. \tag{3.1.5}$$

This model assumes that the average speed is non zero even if the density is the maximal possible; see Figure 3.3.

Moreover one can also consider the Greenshield's model:

$$v(\rho) = v_{max}\left(1 - \left(\frac{\rho}{\rho_{max}}\right)^n\right), \qquad n \in \mathbb{N}, \tag{3.1.6}$$

or the California model:

$$v(\rho) = v_0\left(\frac{1}{\rho} - \frac{1}{\rho_{max}}\right); \tag{3.1.7}$$

see Figures 3.4 and 3.5.

3.1.3 Riemann Problems

Assumptions (LWR1) and (LWR2) imply that equation (3.1.1) is strictly hyperbolic and the characteristic field is genuinely nonlinear. Consider the Rie-

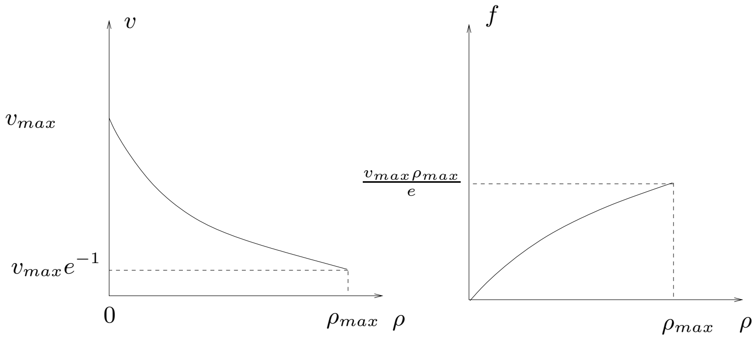


Fig. 3.3. The velocity function and the fundamental diagram when v is given by (3.1.5).

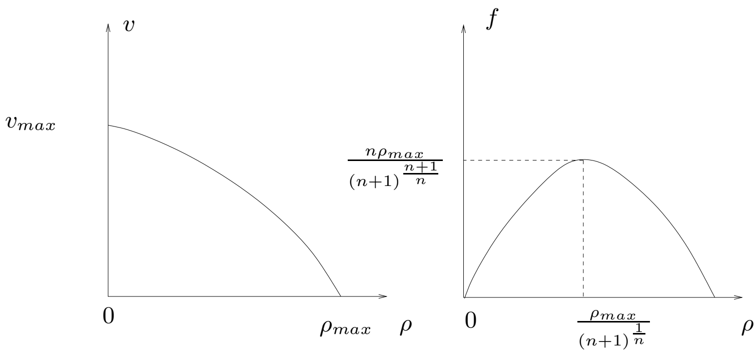


Fig. 3.4. The velocity function and the fundamental diagram when v is given by (3.1.6).

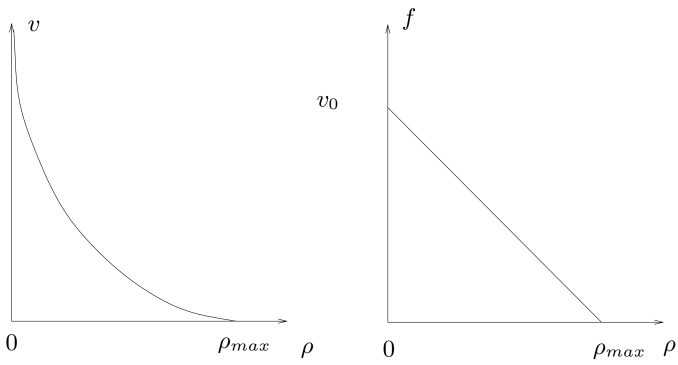


Fig. 3.5. The velocity function and the fundamental diagram when v is given by (3.1.7).

mann problem for (3.1.1) with initial datum

$$\rho(0,x) = \begin{cases} \rho^-, & \text{if } x < 0, \\ \rho^+, & \text{if } x > 0. \end{cases}$$

If $\rho^- < \rho^+$, then (LWR1) and (LWR2) imply that $f'(\rho^-) > f'(\rho^+)$ and so the entropy-admissible solution is given by the shock wave

$$\rho(t,x) = \begin{cases} \rho^-, & \text{if } x < \lambda t, \\ \rho^+, & \text{if } x > \lambda t, \end{cases}$$

where, by the Rankine-Hugoniot condition (2.2.9),

$$\lambda = \frac{f(\rho^+) - f(\rho^-)}{\rho^+ - \rho^-};$$

see Figure 3.6. The speed of the wave is positive if $f(\rho^+) > f(\rho^-)$, while is negative if $f(\rho^+) < f(\rho^-)$.

If instead $\rho^- > \rho^+$, then $f'(\rho^-) < f'(\rho^+)$ and so the entropy-admissible solution to the Riemann problem is given by the rarefaction wave

$$\rho(t,x) = \begin{cases} \rho^-, & \text{if } x < f'(\rho^-)t, \\ (f')^{-1}(\frac{x}{t}), & \text{if } f'(\rho^-)t < x < f'(\rho^+)t, \\ \rho^+, & \text{if } x > f'(\rho^+)t; \end{cases}$$

see Figure 3.7. A particular example of a Riemann problem for this model is given by the description of a traffic light in the introduction.

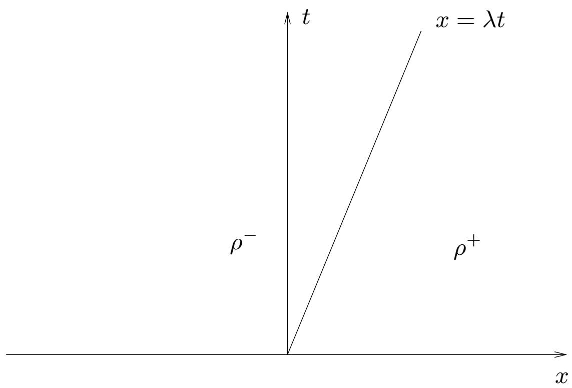


Fig. 3.6. The solution to the Riemann problem when $\rho^- < \rho^+$.

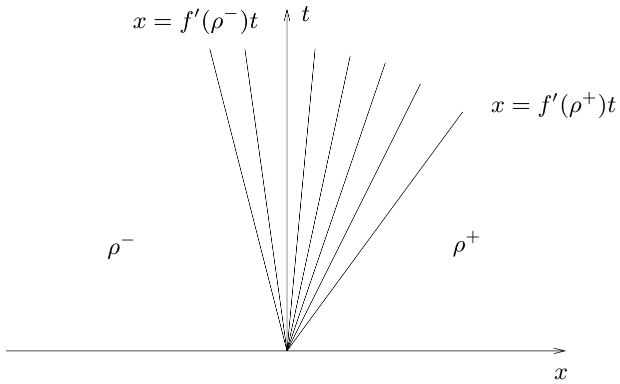


Fig. 3.7. The solution to the Riemann problem when $\rho^- > \rho^+$.

3.1.4 The not Strictly Concave Case

Let us now consider the case in which f is not strictly concave. More precisely, we assume that:

(NSC1) f is a C^2 function;
(NSC2) there exist $0 < \sigma < \theta < 1$ such that the following holds: f is strictly increasing on $[0, \sigma]$ and strictly decreasing on $[\sigma, 1]$; f is strictly concave on $[0, \theta[$ and strictly convex in $]\theta, 1]$; $f'(1) < 0$;
(NSC3) $f(0) = f(1) = 0$.

Again, we can relax (NSC2) at the prize of a more involved treatment.

Let us now illustrate the solutions to Riemann problems. Recalling the construction of Section 2.4.1, define $\bar{\rho}$ to be the point such that the line from $(\bar{\rho}, f(\bar{\rho}))$ to $(1, f(1))$ is tangent to the graph of f at 1; see Figure 3.8. In formulas:

$$\frac{f(1) - f(\bar{\rho})}{1 - \bar{\rho}} = f'(1).$$

Then, similarly, define two functions $\alpha_1 :]\bar{\rho}, \theta[\to]\theta, 1[$ and $\alpha_2 :]\theta, 1] \to]\sigma, \theta[$ in the following way. The secant from $(\alpha_i(\rho), f(\alpha_i(\rho)))$ to $(\rho, f(\rho))$ is tangent to the graph of f at $\alpha_i(\rho)$; see Figure 3.8.
Now, we fix ρ^- and distinguish four cases:

Case 1) $\rho^- \in [0, \bar{\rho}]$;
Case 2) $\rho^- \in]\bar{\rho}, \theta[$;
Case 3) $\rho^- = \theta$;
Case 4) $\rho^- \in]\theta, 1]$.

In Case 1, the solution to Riemann problems with data (ρ^-, ρ^+) is the same as in the strictly concave case.

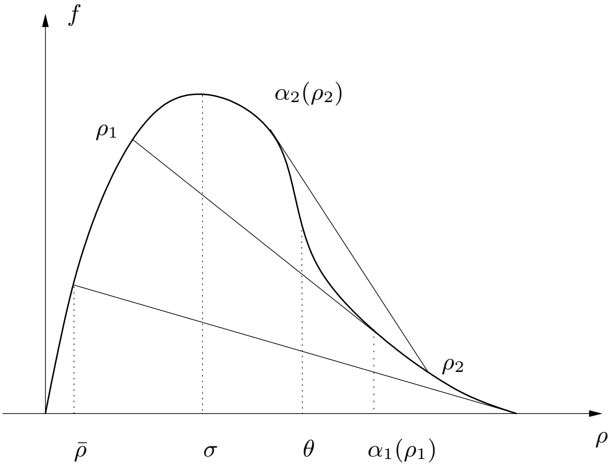


Fig. 3.8. Flux function satisfying (NSC1)–(NSC3) and definition of functions α_1 and α_2.

In Case 2, if $\rho^+ \leqslant \alpha_1(\rho^-)$, then the solution is the same as before, i.e. either a rarefaction ($\rho^+ < \rho^-$) or a shock ($\rho^+ > \rho^-$). If $\rho^+ > \alpha_1(\rho^-)$, then the solution is formed by a shock $(\rho^-, \alpha_1(\rho^-))$ followed by a rarefaction $(\alpha_1(\rho^-), \rho^+)$.

In Case 3, the solution is always a rarefaction.

In Case 4, if $\rho^+ > \rho^-$, then the solution is a rarefaction. If $\alpha_2(\rho^-) \leqslant \rho^+ < \rho^-$, then the solution is a shock. Finally, if $\rho^+ < \alpha_2(\rho^-)$, then the solution is formed by a shock $(\rho^-, \alpha_2(\rho^-))$ followed by a rarefaction $(\alpha_2(\rho^-), \rho^+)$.

3.1.5 Lighthill-Whitham-Richards Model with Viscosity

Some critique to the scalar model are based on the fact that the equation

$$\rho_t + f(\rho)_x = 0$$

develops discontinuities in finite time; see Section 2.2. There is an easy way to eliminate discontinuities in the solution: considering the equation with a viscosity term, i.e.

$$\rho_t + [f(\rho) - \mu\rho_x]_x = 0 \tag{3.1.8}$$

or equivalently

$$\rho_t + f'(\rho)\rho_x = \mu\rho_{xx}, \tag{3.1.9}$$

where μ is a positive constant.

We show here that this equation is not realistic to describe the evolution of the traffic. Consider the following initial-boundary problem: the initial condition is given by

$$\rho_0(x) = \begin{cases} 1, & \text{if } -1 \leqslant x \leqslant 0, \\ 0, & \text{otherwise,} \end{cases} \tag{3.1.10}$$

while the boundary condition $\rho(t, 0) = 1$ holds for every $t \geqslant 0$; see Figure 3.9. Notice that the previous condition corresponds to a traffic light at $x = 0$ with the red light.

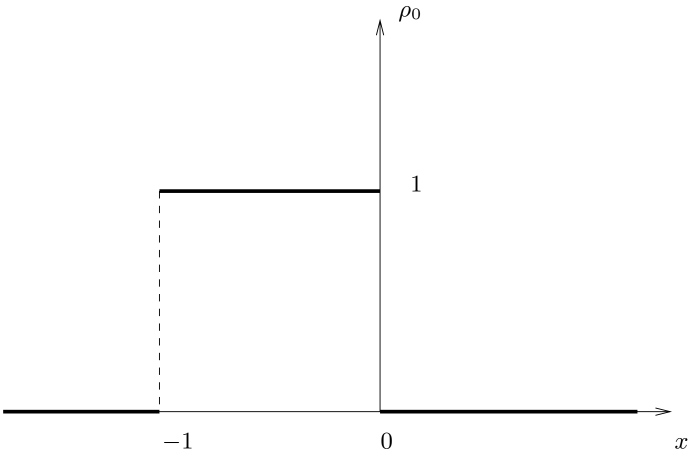


Fig. 3.9. The initial configuration ρ_0.

The natural traffic evolution for this problem should be $\rho(t, x) = \rho_0(x)$ for every $t \geqslant 0$, i.e. nothing changes in time and cars wait at the red light. This is indeed a solution to the inviscid Lighthill-Whitham-Richards model with the boundary condition $\rho(t, 0) = 1$ for every $t \geqslant 0$. Instead, every stationary solution to (3.1.9) satisfies

$$f'(\rho)\rho_x = \mu\rho_{xx}; \tag{3.1.11}$$

thus the function $\rho_0(x)$ is not a stationary solution to (3.1.9). More precisely consider a solution $\bar{\rho}(x)$ to (3.1.11) with boundary conditions

$$\bar{\rho}(0) = 1, \qquad \lim_{x \to -\infty} \bar{\rho}(x) = 0$$

and

$$\int_{-\infty}^{0} \bar{\rho}(x) dx = 1.$$

The previous condition implies that the number of cars before $x = 0$ are the same as at time $t = 0$. Since $\bar{\rho}(x)$ is smooth, we deduce that $\bar{\rho}(x) < 1$ for $x \in [-1, 0[$. Finally, the solution to (3.1.9) tends to $\bar{\rho}(x)$ as $t \to +\infty$ and so some cars move backward, which is completely unrealistic.

3.2 Payne-Whitham Model

In the L-W-R model, the average speed v of cars was supposed to be dependent only on the density ρ. This assumption seems not valid in some interesting traffic flow situations; hence Payne introduced an additional equation for the speed v including a relaxation term for v. This produced the following system of equations:

$$\begin{cases} \rho_t + (\rho v)_x = 0, \\ v_t + v v_x + \frac{1}{\rho}(A_e(\rho))_x = \frac{1}{\tau}(v_e(\rho) - v), \end{cases} \tag{3.2.12}$$

where the term $v_e(\rho)$ is the equilibrium value for the speed,

$$\frac{1}{\rho}(A_e(\rho))_x \tag{3.2.13}$$

is called anticipation term and

$$\frac{1}{\tau}(v_e(\rho) - v) \tag{3.2.14}$$

is called relaxation term of v within a certain time $\tau > 0$ towards its equilibrium value $v_e(\rho)$. The first equation in (3.2.12) is the continuum equation, while the second one is the acceleration equation.

Whitham himself proposed in 1974 a generalization of the L-W-R model adding the following acceleration equation

$$v_t + v v_x + \frac{D}{\rho}\rho_x = \frac{1}{\tau}(v_e(\rho) - v), \tag{3.2.15}$$

with a suitable constant $D > 0$. It is clear that the previous equation is the same of the second equation of (3.2.12) with a particular choice of $A_e(\rho)$. Therefore models with systems (3.2.12) are called Payne-Whitham models.

Various expressions for the term $A_e(\rho)$ have been proposed by some authors. Payne himself supposed that

$$A_e'(\rho) = \frac{1}{2\tau}\left|v_e'(\rho)\right|,$$

while Kühne, Kerner and Kohnhäuser supposed that

$$A_e(\rho) = c_0^2 \rho$$

with a suitable constant c_0. In analogy with fluid dynamics, the term $A_e(\rho)$ can be viewed as "pressure" of traffic.

Sometimes a viscosity term was added in the acceleration equation, which therefore becomes

$$v_t + v v_x + \frac{D}{\rho}\rho_x = \frac{1}{\tau}(v_e(\rho) - v) + \frac{\mu}{\rho} v_{xx}.$$

Remark 3.2.1. As in hydrodynamic case, the term vv_x is called the convection term and describes a motion of the speed profile. The anticipation term (3.2.13) reflects the reaction of identical drivers to the traffic situation in their surroundings. The relaxation term (3.2.14) describes the adaptation of the average speed v to the equilibrium speed $v_e(\rho)$.

Remark 3.2.2. In literature there are also 3x3 models. In particular Helbing proposed a model with 3 equations. The first two equations are for the density and speed, while the third one is for the variance. We refer also to Section 3.5.

3.3 Drawbacks of Second Order Models

Daganzo in 1995 published the paper "Requiem for second-order fluid approximations of traffic flow" about drawbacks of second order models for traffic flow. Payne-Whitham and other second order models are too similar to fluid models, but traffic behaviour presents some important differences with fluid behaviour. The main differences between traffic and fluids are:

1. a fluid particle responds to stimuli from the front and from behind, while a car particle responds only to frontal stimuli;
2. unlike molecules, drivers have personalities.

Daganzo showed in particular that the Payne-Whitham model has this undesirable feature: cars at the end of a queue move backward and this behaviour spreads to the remaining vehicles in the queue.

Consider a traffic light on the red at $x = 0$, as in Section 3.2, with the initial condition

$$\rho_0(x) = \begin{cases} 1, & \text{if } -1 \leqslant x \leqslant 0, \\ 0, & \text{otherwise}, \end{cases} \qquad v_0(\cdot) = 0,$$

and the boundary condition

$$\rho(t,0) = 1, \quad v(t,0) = 0,$$

for every $t \geqslant 0$. The correct solution to this problem should be:

$$\begin{cases} \rho(t,x) = \begin{cases} 0, & \text{if } x < -1, \\ 1, & \text{if } -1 < x < 0, \end{cases} \\ v(t,x) = 0, \quad t > 0, x \leqslant 0. \end{cases} \tag{3.3.16}$$

It means that nothing happens if the traffic light remain on the red. Unfortunately, in general (3.3.16) does not satisfy the stationary the Payne-Whitham model

$$\begin{cases} (\rho v)_x = 0, \\ vv_x + \frac{1}{\rho}(A_e(\rho))_x = \frac{1}{\tau}(v_e(\rho) - v). \end{cases} \tag{3.3.17}$$

As shown in Section 3.2, this implies that $\rho(t,x) < \rho_{max} = 1$ for $x \in [-1, 0[$ and t large enough, so, by the conservation of cars, that there exists a subset I of $(-\infty, -1)$ of positive measure such that $\rho(t,x) > 0$ for every $x \in I$. This again means that some cars travel with negative speed, which is completely unrealistic.

3.4 Aw-Rascle Model

The model is based on the system

$$\begin{cases} \rho_t + (\rho v)_x = 0, \\ (v + p(\rho))_t + v(v + p(\rho))_x = 0, \end{cases} \tag{3.4.18}$$

where $p = p(\rho)$ is the “pressure”, an increasing function of the density. The prototype of function p considered by Aw and Rascle is $p(\rho) = \rho^\gamma$ with $\gamma > 0$.

Multiplying the first equation of (3.4.18) by $p'(\rho)$ and adding to the second one, we obtain

$$\begin{cases} \rho_t + (\rho v)_x = 0, \\ v_t + (v - \rho p'(\rho))v_x = 0. \end{cases} \tag{3.4.19}$$

Clearly, for smooth solutions, systems (3.4.18) and (3.4.19) are equivalent. We have the following proposition.

Proposition 3.4.1. *The system (3.4.19) is hyperbolic. Moreover it is strictly hyperbolic if* $\rho \neq 0$.

Proof. Letting $U := (\rho, v)$, system (3.4.19) can be rewritten in the form

$$U_t + A(U)U_x = 0,$$

where the matrix $A(U)$ is defined by

$$A(U) := \begin{pmatrix} v & \rho \\ 0 & v - \rho p'(\rho) \end{pmatrix}.$$

The eigenvalues of the matrix $A(U)$ are $\lambda_1 = v - \rho p'(\rho)$ and $\lambda_2 = v$. In particular $\lambda_1, \lambda_2 \in \mathbb{R}$. Moreover since $\rho \geqslant 0$ and $p'(\rho) > 0$, we obtain that $\lambda_1 \leqslant \lambda_2$; hence the system (3.4.19) is hyperbolic. If $\rho > 0$, then $\lambda_1 < \lambda_2$ and also the second statement is proved. □

Systems (3.4.18) and (3.4.19) are not in conservation form. In order to obtain a conservative form, let us multiply the first equation of system (3.4.18) by $v + p(\rho)$, the second one by ρ and finally let us add the resulting expression. We obtain:

$$\begin{cases} \rho_t + (\rho v)_x = 0, \\ [\rho(v + p(\rho))]_t + [\rho v(v + p(\rho))]_x = 0. \end{cases} \tag{3.4.20}$$

Therefore if $y := \rho(v + p(\rho))$, then considering the variable $U = (\rho, y)$, system (3.4.20) is in conservation form.

3.4.1 Characteristic Fields

Let us suppose from now on that the pressure function $p(\rho)$ is equal to ρ^γ with $\gamma > 0$. In this case system (3.4.20) can be rewritten in the form

$$\begin{cases} \rho_t + (y - \rho^{\gamma+1})_x = 0, \\ y_t + [\frac{y}{\rho}(y - \rho^{\gamma+1})]_x = 0. \end{cases} \tag{3.4.21}$$

The eigenvalues of the Jacobian matrix of the flux of the system (3.4.21) are given by

$$\lambda_1 = \frac{y}{\rho} - (\gamma+1)\rho^\gamma, \qquad \lambda_2 = \frac{y}{\rho} - \rho^\gamma. \tag{3.4.22}$$

Notice that the second eigenvalue λ_2 is equal to the velocity v of the cars.

It is easy to see that the first characteristic field is genuinely nonlinear, while the second characteristic field is linearly degenerate; see [12, 19]. Moreover the rarefaction curves of the first family are lines passing through the origin. Since the rarefaction curves are lines, also the shock curves of the first family are lines and they have the same expression.

Instead, the curves of the second family through (ρ_0, y_0) are given by

$$y = \frac{y_0}{\rho_0}\rho + \rho^{\gamma+1} - \rho_0^\gamma \rho. \tag{3.4.23}$$

3.4.2 Domains of Invariance

It is natural to assume that the density ρ is positive and bounded by a constant ρ_{max}, which for simplicity we assume to be 1.

Also the velocity v of cars must be positive and bounded. In particular we suppose that the maximum velocity of cars is decreasing with respect to the density ρ and it has the following expression:

$$v_{max}(\rho) = 1 - \rho^\gamma.$$

Thus we obtain that $\rho^{\gamma+1} \leqslant y \leqslant \rho$; see Figure 3.10. Therefore we assume that the variable $U = (\rho, y)$ takes value in the domain

$$\mathcal{D} = \left\{(\rho, y) \in \mathbb{R}^+ \times \mathbb{R}^+ : \rho^{\gamma+1} \leqslant y \leqslant \rho\right\}. \tag{3.4.24}$$

We show that the region $\mathcal{D}$ is invariant for the system (3.4.21). To this purpose, it is enough to show that the solution to every Riemann problem with data in $\mathcal{D}$, remains in $\mathcal{D}$. Consider a road I, modeled by $\mathbb{R}$ and the following Riemann problem:

$$\begin{cases} \partial_t \rho + \partial_x (y - \rho^{\gamma+1}) = 0, \\ \partial_t y + \partial_x (\frac{y^2}{\rho} - y\rho^\gamma) = 0, \\ (\rho(0,x), y(0,x)) = (\rho_-, y_-), & \text{if } x < 0, \\ (\rho(0,x), y(0,x)) = (\rho_+, y_+), & \text{if } x > 0. \end{cases} \tag{3.4.25}$$

There are some different cases.

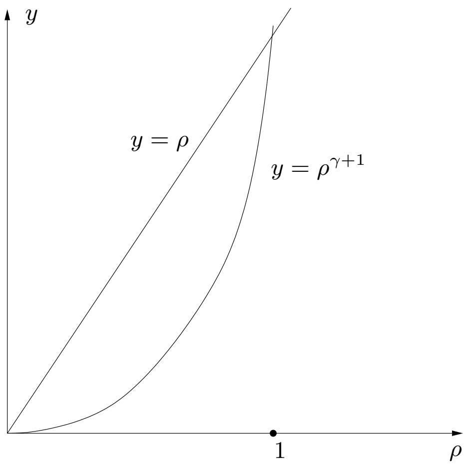


Fig. 3.10. Domain of invariance.

1. The points (ρ_-, y_-) and (ρ_+, y_+) belong either to a curve of the first family or to a curve of the second family. In this case the two points can be connected either by a wave of the first family or by a wave of the second family. Notice that (ρ_-, y_-) or (ρ_+, y_+) can be equal to $(0, 0)$.
2. $\rho_- > 0$, $\rho_+ > 0$ and the curve of the first family through (ρ_-, y_-) intersects the curve of the second family through (ρ_+, y_+) in a point of $\mathcal{D}$ different from $(0, 0)$. We call (ρ_0, y_0) this point.
   If $\rho_0 < \rho_-$ then $\lambda_1(\rho_-, y_-) < \lambda_1(\rho_0, y_0) < \lambda_2(\rho_0, y_0) = \lambda_2(\rho_+, y_+)$. So it is possible to connect (ρ_-, y_-) with (ρ_0, y_0) by a wave of the first family with maximum speed $\lambda_1(\rho_0, y_0)$ and then (ρ_0, y_0) with (ρ_+, y_+) by a wave of the second family with speed $\lambda_2(\rho_0, y_0)$.
   If instead $\rho_0 > \rho_-$, then it is possible to connect (ρ_-, y_-) with (ρ_0, y_0) by a shock wave of the first family with speed
   $$\frac{(y_- - \rho_-^{\gamma+1}) - (y_0 - \rho_0^{\gamma+1})}{\rho_- - \rho_0}$$
   and then (ρ_0, y_0) with (ρ_+, y_+) by a wave of the second family with speed
   $$\lambda_2(\rho_0, y_0) = \frac{y_0}{\rho_0} - \rho_0^\gamma.$$
   Clearly this process is admissible if and only if
   $$\frac{(y_- - \rho_-^{\gamma+1}) - (y_0 - \rho_0^{\gamma+1})}{\rho_- - \rho_0} < \frac{y_0}{\rho_0} - \rho_0^\gamma. \tag{3.4.26}$$
   Since (ρ_-, y_-) and (ρ_0, y_0) belong to the same line $y = c\rho$ with $c > 0$, (3.4.26) is valid if and only if

$$\frac{c(\rho_- - \rho_0) - (\rho_-^{\gamma+1} - \rho_0^{\gamma+1})}{\rho_- - \rho_0} < c - \rho_0^\gamma$$

which is equivalent to

$$\frac{\rho_-^{\gamma+1} - \rho_0^{\gamma+1}}{\rho_- - \rho_0} > \rho_0^\gamma.$$

Multiplying by $(\rho_- - \rho_0)$ the last inequality, it results $\rho_-^{\gamma+1} - \rho_0^{\gamma+1} < \rho_0^\gamma \rho_- - \rho_0^{\gamma+1}$ and so (3.4.26) is equivalent to $\rho_-^\gamma < \rho_0^\gamma$ which is clearly true. Thus the analysis of this case is completed.

3. $\rho_- > 0, \rho_+ > 0$ and the curve of the first family through (ρ_-, y_-) intersects in $\mathcal{D}$ the curve of the second family through (ρ_+, y_+) only at $(0,0)$. Let $y = c_1\rho$ be the curve of the first family through (ρ_-, y_-) and let $y = c_2\rho + \rho^{\gamma+1}$ be the curve of the second family through (ρ_+, y_+). In this case it is easy to see that $c_1 \leqslant c_2$. It is possible to connect (ρ_-, y_-) to $(0,0)$ by a wave of the first family whose maximum speed is

$$\lim_{\rho \to 0^+} \lambda_1(\rho, c_1\rho) = \lim_{\rho \to 0^+} c_1 - (\gamma+1)\rho^\gamma = c_1$$

and then $(0,0)$ to (ρ_+, y_+) by a wave of the second family with speed c_2. The conclusion follows from the fact that $c_2 \geqslant c_1$.

Remark 3.4.2. It is clear by the previous analysis that, once we consider the domain $\mathcal{D}$, cars have always positive speed. In fact, each point (ρ, y) in $\mathcal{D}$ satisfies

$$v = \frac{y}{\rho} - \rho^\gamma \geqslant 0.$$

Moreover, since $\mathcal{D}$ is invariant under the solution to the Riemann problem and hence to the Cauchy problem, the function $(\rho(t,x), y(t,x)) \in \mathcal{D}$, i.e. cars have always positive speed.

3.5 Third Order Models

The first third order model was proposed in 1995 by Dirk Helbing. He considered non only equations for density and velocity, but also for the variance θ. The last variable becomes important to describe and predict traffic jams. In fact fast increment of the variance describes queue formation in car traffic.

The exact model proposed by Helbing is the following one.

$$\begin{cases} \rho_t + (\rho v)_x = 0, \\ v_t + v v_x + \frac{1}{\rho}(\rho\theta)_x = \frac{1}{\tau}(v_e(\rho) - v) + \frac{\mu}{\rho} v_{xx}, \\ \theta_t + v\theta_x + 2\theta v_x = 2\frac{\mu}{\rho}(v_x)^2 + \frac{k}{\rho}\theta_{xx} + \frac{2}{\tau}(\theta_e(\rho) - \theta), \end{cases} \tag{3.5.27}$$

where θ_e and v_e are given smooth functions of the density ρ, while μ, k, τ are positive constants. The coefficient k is called *kinetic coefficient*. The quantity

$$J(t,x) := -k\theta_x$$

describes a flux of velocity variance leading to a spatial smoothing of θ. The term originates from the finite reaction and braking time, which causes a delayed adaption of speed to traffic situation. The term

$$\frac{2}{\tau}(\theta_e(\rho) - \theta)$$

results from the drivers' attempt to drive with their desired velocities and from drivers' interactions, i.e. from deceleration in a situation when a fast car can not overtake a slower one.

In analogy with gas dynamics, the previous model is called an Euler type model if μ and k are both equal to 0, otherwise it is called a Navier Stokes type model. Stability analysis and numerical simulations about this model can be found in [61].

3.6 Hyperbolic Phase Transition Model

A hyperbolic phase transition model for traffic was introduced by Colombo in 2002. Two phases corresponding to the free and congested flow are considered. In the free flow the Lighthill-Whitham-Richards equation

$$\rho_t + (v\rho)_x = 0 \tag{3.6.28}$$

is assumed to hold; see Section 3.1.

When the car density ρ is high, the assumption that the speed v is a function only of the density is no longer taken. As soon as the speed v crosses a certain value, the density-flow points are scattered in a two-dimensional region. Thus in the congested region the traffic evolution follows the equation

$$\begin{cases} \partial_t\rho + \partial_x\left[\rho \cdot v\right] = 0, \\ \partial_t q + \partial_x\left[(q-Q)\cdot v\right] = 0, \end{cases} \tag{3.6.29}$$

where v is a known function depending on the density ρ and on the momentum q, while Q is a given parameter. From the traffic point of view, the parameter Q is characterized by the phenomenon of *wide jams*; see [29].

The complete model is described by

$$\begin{array}{cc} \textbf{Free flow} & \textbf{Congested flow} \\ \begin{cases} (\rho,q) \in \Omega_f, \\ \rho_t + \left[\rho\cdot v\right]_x = 0, \\ v = \left(1-\frac{\rho}{R}\right)\cdot V, \end{cases} & \begin{cases} (\rho,q)\in\Omega_c, \\ \rho_t + \left[\rho\cdot v\right]_x = 0, \\ q_t + \left[(q-Q)\cdot v\right]_x = 0, \\ v = \left(1-\frac{\rho}{R}\right)\cdot\frac{q}{\rho}\,. \end{cases} \end{array} \tag{3.6.30}$$

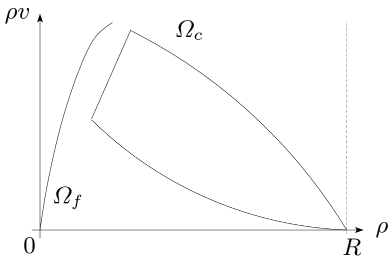


Fig. 3.11. The fundamental diagram for (3.6.30).

Here R and V are respectively the maximal vehicle density and speed.

The *free* and the *congested* phase Ω_f and Ω_c (see Figure 3.11) are respectively defined by

$$\Omega_f = \{(\rho, q) \in [0, R] \times [0, +\infty[\, : v_f(\rho) \geqslant V_f, \, q = \rho \cdot V\} \tag{3.6.31}$$

and

$$\Omega_c = \left\{ (\rho, q) \in [0, R] \times [0, +\infty[: v_c(\rho, q) \leqslant V_c, \, \frac{Q^- - Q}{R} \leqslant \frac{q - Q}{\rho} \leqslant \frac{Q^+ - Q}{R} \right\}, \tag{3.6.32}$$

where V_f and V_c are the threshold speeds, i.e. above V_f the flow is free, while below V_c the flow is congested, and the parameters $Q^- \in]0, Q[$, $Q^+ \in]Q, +\infty[$ depend on environmental conditions and determine the width of the region Ω_c. It is assumed that the various parameters are strictly positive and satisfy

$$V > V_f > V_c, \quad Q^+ \geqslant Q \geqslant Q^-, \quad \frac{Q^+ - Q}{RV} < 1 \tag{3.6.33}$$

and

$$V_f = \frac{V - Q^+/R}{1 - (Q^+ - Q)/(RV)} \, . \tag{3.6.34}$$

Finally, we recall the condition

$$\left(1 - \frac{Q^+}{RV}\right) \cdot \left(\frac{Q^+}{Q} - 1\right) < 1 \, , \tag{3.6.35}$$

that ensures that all the waves of the first family have negative speed; see [37, Proposition 2.3].

3.6.1 The Riemann Problem

Here we discuss about the Riemann problem in the phase transition traffic model. First let us analyze the characteristic speeds and fields in the congested phase, since the free phase was treated in Section 3.1. Simple computations show that the characteristic speeds in the congested phases are given by

$$\lambda_1(\rho, q) = \left(\frac{2}{R} - \frac{1}{\rho}\right)(Q - q) - \frac{Q}{R},$$

and

$$\lambda_2(\rho, q) = v_c(\rho, q) \geqslant 0.$$

If condition (3.6.35) holds, then we have that $\lambda_1 \leqslant 0$; see [28]. Moreover we have that the first characteristic field is genuinely nonlinear, while the second one is linearly degenerate. The curve of the first family exiting (ρ_0, q_0) is a line given by

$$q = Q + \frac{q_0 - Q}{\rho_0}\,\rho,$$

while the curve of the second family exiting (ρ_0, q_0) is

$$q = \frac{\rho}{\rho_0}\,\frac{R - \rho_0}{R - \rho}\,q_0;$$

see Figure 3.12.

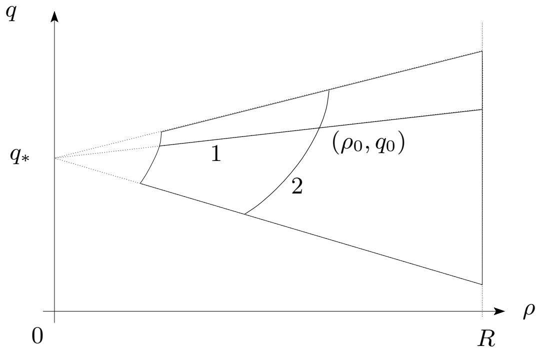


Fig. 3.12. The Lax curves exiting from (ρ_0, q_0) in the congested region.

So, let us consider the initial datum

$$(\rho, q)(0, x) = \begin{cases} (\rho_l, q_l), & \text{if } x < 0, \\ (\rho_r, q_r), & \text{if } x > 0. \end{cases} \tag{3.6.36}$$

If (ρ_l, q_l) and (ρ_r, q_r) belong to the same phase, then the classical solution is used. Hence the significant case is when $(\rho_l, q_l) \in \Omega_f$ and $(\rho_r, q_r) \in \Omega_c$ or viceversa. The definition of solution in this case is the following one.

Definition 3.6.1. *Consider a Riemann problem where the initial states* (ρ_l, q_l) *and* (ρ_r, q_r) *belong to different phases. An admissible solution is a self similar function*

$$u : [0, +\infty[\times \mathbb{R} \to \Omega_f \cup \Omega_c$$

such that there exists $\bar{\lambda} \in \mathbb{R}$ *satisfying:*

1. *if* $(\rho_l, q_l) \in \Omega_f$ *and* $(\rho_r, q_r) \in \Omega_c$, *then* $u(t,]-\infty, \bar{\lambda}t[) \subseteq \Omega_f$ *and* $u(t,]\bar{\lambda}t, +\infty[) \subseteq \Omega_c$ *for every* $t > 0$;
2. *if* $(\rho_l, q_l) \in \Omega_c$ *and* $(\rho_r, q_r) \in \Omega_f$, *then* $u(t,]-\infty, \bar{\lambda}t[) \subseteq \Omega_c$ *and* $u(t,]\bar{\lambda}t, +\infty[) \subseteq \Omega_f$ *for every* $t > 0$;
3. *the functions*
$$u_l(t,x) = \begin{cases} u(t,x), & \text{if } x < \bar{\lambda}t, \\ u(t, \bar{\lambda}t-), & \text{if } x > \bar{\lambda}t, \end{cases}$$
and
$$u_r(t,x) = \begin{cases} u(t, \bar{\lambda}t+), & \text{if } x < \bar{\lambda}t, \\ u(t,x), & \text{if } x > \bar{\lambda}t, \end{cases}$$
are solutions to the Riemann problems with initial data respectively given by
$$\begin{cases} (\rho_l, q_l), & \text{if } x < 0, \\ u(t, \bar{\lambda}t-), & \text{if } x > 0, \end{cases}$$
and
$$\begin{cases} u(t, \bar{\lambda}t+), & \text{if } x < 0, \\ (\rho_r, q_r), & \text{if } x > 0; \end{cases}$$
4. *if* $(\rho_l, q_l) \in \Omega_f$ *and* $(\rho_r, q_r) \in \Omega_c$, *then the Rankine-Hugoniot condition*
$$\bar{\lambda}\left(u(t, \bar{\lambda}t+) - u(t, \bar{\lambda}t-)\right) = u(t, \bar{\lambda}t+) v_c(t, \bar{\lambda}t+) - u(t, \bar{\lambda}t-) v_f(t, \bar{\lambda}t-)$$
holds;
5. *if* $(\rho_l, q_l) \in \Omega_c$ *and* $(\rho_r, q_r) \in \Omega_f$, *then the Rankine-Hugoniot condition*
$$\bar{\lambda}\left(u(t, \bar{\lambda}t+) - u(t, \bar{\lambda}t-)\right) = u(t, \bar{\lambda}t+) v_f(t, \bar{\lambda}t+) - u(t, \bar{\lambda}t-) v_c(t, \bar{\lambda}t-)$$
holds.

Given two initial states (ρ_l, q_l) and (ρ_r, q_r) belonging to different phases, there exists a unique $\bar{\lambda} \in \mathbb{R}$ and an admissible solution, which is constant in the regions
$$\left\{(t,x) \in \mathbb{R}^2 : t \geqslant 0, x < \bar{\lambda}t\right\}$$
and
$$\left\{(t,x) \in \mathbb{R}^2 : t \geqslant 0, x > \bar{\lambda}t\right\}.$$

3.7 A Multilane Model

Here we present a multilane model, which is an extension of the L-W-R model. The main novelty is considering an unidirectional one-dimensional road with n lanes.

The macroscopic variables considered here are the density ρ of cars and the average speed v across all the lanes. Thus we have

$$\rho = \sum_{i=1}^{n} \rho_i,$$

and

$$\rho v = \sum_{i=1}^{n} \rho_i v_i,$$

where ρ_i and v_i are respectively the density and the average speed of cars in the i-th lane.

In a multilane road, one can often observe different traffic behaviour depending on the density of traffic. When traffic is low, changing lane and overcome cars is easy and so the equilibrium speed for cars is high. When traffic is high, these actions become complicate and difficult, so that the equilibrium speed for cars is low. The typical situation involves two distinct equilibria for the average speed of cars. This is described by two functions $w_1(\rho)$ and $w_2(\rho)$ defined on $[0, \rho_{max}]$ such that

$$w_1(\rho) > w_2(\rho)$$

for every $\rho \in [0, \rho_{max}[$ and $w_1(\rho_{max}) = w_2(\rho_{max}) = 0$. When the density ρ is less than a critical value $\bar{\rho}_1$, then the average speed is described by the function w_1, while when the density ρ is greater than a value $\bar{\rho}_2$, then the average speed is described by the function w_2; see Figure 3.13.

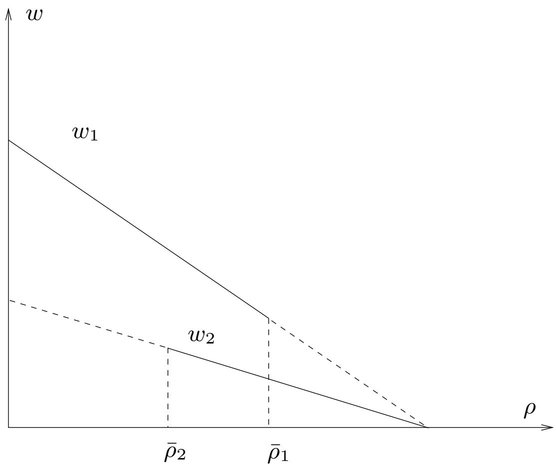


Fig. 3.13. The functions w_1 and w_2 for the multilane model.

Defining $\alpha = v - w_1(\rho)$ the system is given by

$$\begin{cases} \rho_t + (\rho v)_x = 0, \\ \alpha_t + v\alpha_x = \begin{cases} -\frac{\alpha}{\varepsilon}, & \rho < R(v), \\ \frac{(w_2(\rho) - w_1(\rho)) - \alpha}{\varepsilon}, & \rho \geqslant R(v), \end{cases} \end{cases} \tag{3.7.37}$$

where $R(v)$ is a monotone non-decreasing function defined on $\mathbb{R}^+$ satisfying

$$R(v) = \bar{\rho}_2, \quad \forall 0 \leqslant v \leqslant w_2(\bar{\rho}_2)$$

and

$$R(v) = \bar{\rho}_1, \quad \forall v \geqslant w_1(\bar{\rho}_1),$$

and ε is a small positive constant.

Notice that (3.7.37) in terms of ρ and v becomes

$$\begin{cases} \rho_t + (\rho v)_x = 0, \\ v_t + (v + \rho w_1'(\rho))v_x = \begin{cases} \frac{w_1(\rho) - v}{\varepsilon}, & \rho < R(v), \\ \frac{w_2(\rho) - v}{\varepsilon}, & \rho \geqslant R(v). \end{cases} \end{cases} \tag{3.7.38}$$

Remark 3.7.1. The system (3.7.37) is nothing else that the Aw-Rascle model with a source term depending by the density of the road. In fact, without source term, (3.7.37) takes the form

$$\begin{cases} \rho_t + (\rho v)_x = 0, \\ \alpha_t + v\alpha_x = 0, \end{cases}$$

which is completely equivalent to (3.4.18) with the position

$$p(\rho) = -w_1(\rho) + c,$$

for every positive constant c.

3.8 A Multipopulation Model

Multipopulation models are extensions of the LWR model and their aim is to predict the behaviour of different heterogeneous drivers. Some models of this kind were proposed independently by Wong and Wong in 2002 and by Benzoni-Gavage and Colombo in 2003.

The one for n-populations proposed by Benzoni-Gavage and Colombo can be written in the form

$$\partial_t \rho_i + \partial_x(\rho_i v_i) = 0, \quad i = 1, \ldots, n, \tag{3.8.39}$$

where ρ_i is the density of cars belonging to the i-th class (or population) of drivers and v_i is the average speed of the i-th family and it is a function depending on $(\rho_1, \ldots, \rho_n)$. Notice that (3.8.39) is a system of n conservation laws. Without additional hypotheses on the functions v_i, (3.8.39) is not in general hyperbolic. Thus we consider the following assumptions:

(A1) For every $i \in \{1, \dots, n\}$, the function v_i depends only on $r = \rho_1 + \cdots + \rho_n$.

(A2) There exists a scalar function $\psi(r) \in [0,1]$ such that, for every $i \in \{1, \dots, n\}$,

$$v_i(r) = \psi(r) V_i, \tag{3.8.40}$$

where V_i is the maximal speed for the i-th population.

The expression for ψ is given by the fundamental diagram. For notational simplicity we rescale the system so that $r_{max} = 1$. In this case the model (3.8.39) is defined in the domain

$$\mathcal{D} = \{(\rho_1, \dots, \rho_n) \in \mathbb{R}^n : \rho_i \geqslant 0 \text{ and } \rho_1 + \cdots + \rho_n \leqslant 1\}. \tag{3.8.41}$$

Setting $U = (\rho_1, \dots, \rho_n)$ and

$$f(U) = \psi(r)(\rho_1 V_1, \dots, \rho_n V_n),$$

the system (3.8.39) takes the form

$$U_t + f(U)_x = 0. \tag{3.8.42}$$

Proposition 3.8.1. *If $\rho_i > 0$ for every $i \in \{1, \dots, n\}$, then (3.8.42) is symmetrisable and hyperbolic. Moreover*

$$E(\rho_1, \dots, \rho_n) = \sum_{i=1}^{n} \frac{\rho_i(\ln \rho_i - 1)}{V_i}$$

is a convex entropy and

$$F(\rho_1, \dots, \rho_n) = \psi(r) \sum_{i=1}^{n} \rho_i \ln \rho_i - \Psi(r)$$

is a corresponding entropy flux, where Ψ is a primitive of ψ.

Proof. The Jacobian matrix of the flux f is

$$\begin{pmatrix} \psi(r)V_1 + \psi'(r)\rho_1 V_1 & \cdots & \psi'(r)\rho_1 V_1 \\ \vdots & \ddots & \vdots \\ \psi'(r)\rho_n V_n & \cdots & \psi(r)V_n + \psi'(r)\rho_n V_n \end{pmatrix}.$$

If $\rho_i > 0$ for every $i \in \{1, \dots, n\}$, then the diagonal matrix

$$\begin{pmatrix} \frac{1}{\rho_1 V_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \frac{1}{\rho_n V_n} \end{pmatrix}.$$

is a symmetriser for Df, i.e. $S \cdot Df$ is symmetric. This implies that (3.8.42) is hyperbolic; see [98].

Concerning the entropy, it is sufficient to note that

$$\partial_t E(U) + \partial_x F(U) = 0,$$

and this concludes the proof. $\square$

3.8.1 The Case $n = 2$

Consider now the special case of the model with just 2 populations, i.e. $n = 2$. Using the expression $1 - r$ for the function ψ, the model reduces to the system of two equation

$$\begin{cases} \partial_t \rho_1 + \partial_x (V_1 (1 - \rho_1 - \rho_2) \rho_1) = 0, \\ \partial_t \rho_2 + \partial_x (V_2 (1 - \rho_1 - \rho_2) \rho_2) = 0. \end{cases} \tag{3.8.43}$$

Assume that $V_1 > V_2$. The following proposition holds.

Proposition 3.8.2. *The system (3.8.43) is hyperbolic in the domain*

$$\mathcal{D} = \{(\rho_1, \ldots, \rho_n) \in \mathbb{R}^n : \rho_i \geqslant 0 \ \text{ and } \rho_1 + \cdots + \rho_n \leqslant 1\}.$$

At the point $P := (\frac{V_1 - V_2}{2V_1 - V_2}, 0)$ the eigenvalues of the Jacobian matrix coalesce. In $\mathcal{D} \setminus P$, the system is strictly hyperbolic.

Proof. The Jacobian matrix for the flux of the system (3.8.43) is

$$\begin{pmatrix} V_1(1 - \rho_1 - \rho_2) - \rho_1 V_1 & -\rho_1 V_1 \\ -\rho_2 V_2 & V_2(1 - \rho_1 - \rho_2) - \rho_2 V_2 \end{pmatrix}.$$

The characteristic polynomial of the Jacobian matrix is

$$(\lambda - V_1(1 - 2\rho_1 - \rho_2))\,(\lambda - V_2(1 - \rho_1 - 2\rho_2)) - V_1 V_2 \rho_1 \rho_2$$

and its discriminant is

$$(V_1(1 - 2\rho_1 - \rho_2) - V_2(1 - \rho_1 - 2\rho_2))^2 + 4V_1 V_2 \rho_1 \rho_2,$$

which is greater then or equal to 0. Hence the system is hyperbolic. Moreover the discriminant is equal to 0 if and only if

$$\rho_1 \rho_2 = 0, \quad V_1(1 - 2\rho_1 - \rho_2) = V_2(1 - \rho_1 - 2\rho_2).$$

If $\rho_1 = 0$, then the previous conditions read

$$(V_1 - V_2)(1 - \rho_2) = -V_2 \rho_2.$$

The first term is positive, while the second term negative; hence we reach a contradiction.

If $\rho_2 = 0$, then the conditions read

$$(V_1 - V_2)(1 - \rho_1) = V_1 \rho_1,$$

and so

$$\rho_1 = \frac{V_1 - V_2}{2V_1 - V_2}.$$

This concludes the proof. □

It is also possible to prove that the first characteristic field is genuinely nonlinear, while the second characteristic field satisfies

$$r_2 \bullet \lambda_2 = 0$$

if and only if $\rho_1 + \rho_2 = 1$, where λ_2 and r_2 denote respectively the second eigenvalue and eigenfunction. For a proof of this fact see [16].

3.9 Exercises

Exercise 3.9.1. Consider the Lighthill-Whitham-Richards model for traffic when the flux f is given by

$$f(\rho) = \rho(1 - \rho).$$

Find the exact solution to the Cauchy problem

$$\begin{cases} \rho_t + f(\rho)_x = 0, \\ \rho(0, x) = \rho_0(x), \end{cases}$$

when the the initial datum ρ_0 has the following expressions:

1.
$$\rho_0(x) = \begin{cases} \frac{1}{4}, & \text{if} \quad x < 0, \\ \frac{1}{2}, & \text{if} \quad 0 < x < 1, \\ \frac{1}{4}, & \text{if} \quad x > 1; \end{cases}$$

2.
$$\rho_0(x) = \begin{cases} \frac{1}{4}, & \text{if} \quad x < 0, \\ \frac{3}{4}, & \text{if} \quad 0 < x < 1, \\ \frac{1}{2}, & \text{if} \quad x > 1; \end{cases}$$

3.
$$\rho_0(x) = \frac{1}{\pi} \left[\frac{\pi}{2} - \arctan x \right].$$

Exercise 3.9.2. Find the solutions to the Cauchy problems of the previous exercise, with the Underwood flux

$$f(\rho) = \rho e^{-\rho}$$

and with the California model flux

$$f(\rho) = \frac{1}{\rho} - 1.$$

Exercise 3.9.3. Find solutions, if any, defined on $]-\infty, 0]$ to equation (3.1.11) such that $\rho(0) = 1$, $\lim_{x \to -\infty} = c$ for the flux $f(\rho) = \rho(1 - \rho)$.
Hint: Write solutions as trajectories on the phase space $\theta = \rho$ and $\dot{\theta} = \rho_x$ for a suitable vector field.

Exercise 3.9.4. Consider the Aw-Rascle system (3.4.21) and assume that $v_{max}(\rho) \equiv 1$.

1. Is the region $\{(\rho, y) : 0 \leq \rho \leq \rho_{max}, 0 \leq v \leq v_{max}\}$ invariant for (3.4.21)?
2. Is $\mathcal{D}$, given by (3.4.24), still an invariant region?

Exercise 3.9.5. Consider the phase transition model of Section 3.6. For a fixed density ρ_l giving the datum in the free phase, find all data (ρ_r, q_r) in the congested phase, which can be connected to ρ_l by a single wave, i.e. a phase transition, with positive velocity.

3.10 Bibliographical Note

The first order model described in Section 3.1 was proposed independently by Lighthill and Whitham [88] in 1955 and by Richards [95] in 1956. For a theoretical study of the LWR model, it is fundamental to know expressions of fundamental diagrams. A comparison between fundamental diagrams can be founded in [48, 89]. As regards the Greenberg's fundamental diagram, one may refer to [54].

Kerner in [71, 72, 74] affirmed that essentially a fundamental diagram does not exists, since he observed that traffic flow presents three different behaviors, depending on whether it is free or congested; see also [73].

The case of not concave fundamental diagrams was treated by some authors; see for example Helbing [61, 62], Sopasakis and Katsoulakis [100] and Tong Li [102]. Some complicated traffic flow patterns produced in the case of not concave fundamental diagrams can be found in Greenberg [55], Greenberg, Klar and Rascle [57], Helbing [61, 62], Kerner and Konhäuser [75] and Tong Li [104].

The Payne-Whitham model was proposed independently by Payne [93, 92] in 1971 and by Whitham [107] in 1974. Some variants were proposed by Philips [94] in 1977, by Kühne [81] in 1991 and by Kerner and Konhäuser [75] in 1994.

The paper by Daganzo [39] in 1995 stopped the growth of second order models showing bad behaviours of cars in these models.

Aw and Rascle [12] in 2000 proposed a second order model taking care of all the observations in the paper by Daganzo. A similar second order model for traffic was also proposed by Zhang [110]. Aw, Klar, Materne and Rascle in [11] gave a derivation for the Aw-Rascle model. They showed that this model can be viewed as the limit of some *follow the leader* models.

The first third order model was proposed by Helbing in 1995; see [61].

According to Kerner's considerations, Colombo [28, 29] in 2002 proposed an hyperbolic phase transition model; see also [30]. The well posedeness of the phase transition model was done in [37].

A multilane extension of the Aw-Rascle model was proposed by Greenberg, Klar and Rascle [57], while the multipopulation model, proposed in this book, was introduced independently by Wong and Wong [109] and by Benzoni–Gavage and Colombo in 2003 [16]. In the literature, there are some other multilane and multipopulation models; see for example Hoogendoorn [68] or Herman and Prigogine [63].

There are also some traffic models describing bottlenecks, entries, exits and some other realistic situations. One may refer to Hoogendoorn [68], to May [89] and to Daganzo [38].

4

Networks

This chapter deals with a complex network represented by a directed graph, that is a finite collection of directed edges, connected together at some vertices. Each vertex is given by a finite number of incoming edges and of outgoing edges. We describe the general approach to define a system on the whole network, by determining the behavior at vertices. In next chapters, on each edge we consider a model for traffic evolution.

4.1 Basic Definitions and Assumptions

First we introduce the definition of network.

Definition 4.1.1. *A network is a couple* $(\mathcal{I}, \mathcal{J})$ *where:*

$\mathcal{I}$ *is a finite collection of edges, which are intervals in* $\mathbb{R}$, $I_i = [a_i, b_i] \subseteq \mathbb{R}$, $i = 1, \ldots, N$;

$\mathcal{J}$ *is a finite collection of vertices. Each vertex* J *is union of two non empty subsets* $\mathrm{Inc}(J)$ *and* $\mathrm{Out}(J)$ *of* $\{1, \ldots, N\}$.

We assume the following:

1. *For every* $J \neq J' \in \mathcal{J}$ *we have* $\mathrm{Inc}(J) \cap \mathrm{Inc}(J') = \emptyset$ *and* $\mathrm{Out}(J) \cap \mathrm{Out}(J') = \emptyset$.
2. *If* $i \notin \cup_{J \in \mathcal{J}} \mathrm{Inc}(J)$ *then* $b_i = +\infty$ *and if* $i \notin \cup_{J \in \mathcal{J}} \mathrm{Out}(J)$ *then* $a_i = -\infty$. *Moreover, the two cases are mutually exclusive.*

The two conditions essentially asks the network to be graph. According to the previous definition, each vertex can be identified with a $n + m$–tuple $(i_1, \ldots, i_n, i_{n+1}, \ldots, i_{n+m})$, where the first n-tuple indicates the set of incoming edges and the second m-tuple indicates the set of outgoing edges. Condition 1 implies that each edge can be incoming for at most one vertex and outgoing for at most one vertex. Moreover, condition 2 implies that some edges may extend to infinity but are connected to at least one vertex; see Figure 4.1.

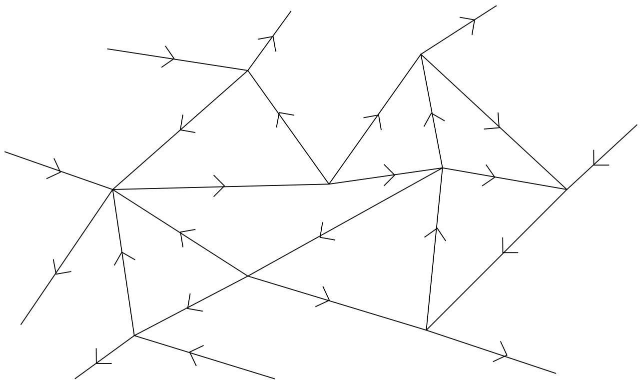


Fig. 4.1. Example of network.

4.2 Riemann Solvers

In this section, assuming that the traffic on each edge is represented by an hyperbolic system of conservation laws:

$$(u_i)_t + (f_i(u_i))_x = 0, \qquad u_i \in \mathbb{R}^p, \tag{4.2.1}$$

we want to define and solve Riemann problems at vertices. So we fix a network $(\mathcal{I}, \mathcal{J})$ and a vertex $J \in \mathcal{J}$ and assume that $\mathrm{Inc}(J) = \{1, \dots, n\}$ and $\mathrm{Out}(J) = \{n+1, \dots, n+m\}$.

Definition 4.2.1. *A Riemann problem at J is a Cauchy problem corresponding to an initial data which is constant on each edge.*

We look for centered solutions on each edge, which are the building blocks to construct solutions to the Cauchy problem via a wave-front tracking algorithm. Thus on each edge, we expect a solution to be formed by rarefactions, shocks or contact discontinuities.

A Riemann solver is a map assigning a solution to each Riemann initial data. Since we consider only centered solutions, it is sufficient to assign on each edge a constant boundary value.

Definition 4.2.2. *A Riemann Solver for the vertex J is a function*

$$RS : (\mathbb{R}^p)^{n+m} \to (\mathbb{R}^p)^{n+m}$$

that associates to every Riemann data $u_0 = (u_{1,0}, \dots, u_{n+m,0})$ at J a vector $\hat{u} = (\hat{u}_1, \dots, \hat{u}_{n+m})$ so that the following holds.
On each edge I_i, $i = 1, ..., n+m$, the solution is given by the solution to the initial–boundary value problem with initial data $u_{0,i}$ and boundary data $\hat{u}_i$.
We require the consistency condition

(CC) $RS(RS(u_0)) = RS(u_0)$.

Remark 4.2.3. A more general definition is obtained if the map RS depends also on some parameters, say π, which are constant or evolve in time. This is precisely the case of Chapter 7.

Once a Riemann solver is assigned we can define admissible solutions at J.

Definition 4.2.4. *Assume a Riemann Solver RS is assigned at junction J. Let* $u = (u_1, \dots, u_{n+m})$*,* $u_i : [0, +\infty) \times I_i \to \mathbb{R}^p$ *be such that* $u_i(t, \cdot)$ *is of bounded variation for every* $t \geq 0$*. Then u is an admissible weak solution to (4.2.1) related to RS at the vertex J if and only if the following properties hold:*

(i) u_i *is a weak solution to (4.2.1) on the edge;*
(ii) for almost every t, setting

$$u_J(t)=(u_1(\cdot,b_1-),\dots,u_n(\cdot,b_n-),u_{n+1}(\cdot,a_{n+1}+),\dots,u_{n+m}(\cdot,a_{n+m}+)),$$

we have

$$RS(u_J(t)) = u_J(t).$$

Such a general definition includes various "non–physical" cases. For example, in various applications, the quantity u, or some components of u, must be conserved also at the vertex J, i.e. at the vertex J there is no creation or destruction of u. This means that the total flowing in traffic must be equal to the total flowing out traffic at J. One necessary condition is to ask equality of incoming and outgoing fluxes for the obtained solution $\hat{u}$. However, such condition is not sufficient. Indeed, the initial-boundary value problem on each edge may produce a solution which does not attain the boundary value pointwise. So, to ensure conservation of u, we must ask for solutions to the initial boundary value problems to have negative characteristic velocities on incoming edges and positive characteristic velocities on outgoing ones. This amounts exactly to ask that the Riemann problem on a real line with initial data $(u_{i,0}, \hat{u}_i)$, $i = 1, \dots, n$, produces only waves with negative velocities and the Riemann problem on a real line with initial data $(\hat{u}_j, u_{j,0})$, $j = n+1, \dots, n+m$, produces only waves with positive velocities. Finally, conservation of u at the vertex J is equivalent to ask:

(Cons.1) If $\hat{u} = RS(u_0)$, then for incoming edges the solution to the Riemann problem $(u_{i,0}, \hat{u}_i)$ has all waves with strictly negative speed, $i = 1, \dots, n$, while for outgoing edges the solution to the Riemann problem $(\hat{u}_j, u_{j,0})$ has all waves with strictly positive speed, $j = n+1, \dots, n+m$.

(Cons.2) If $\hat{u} = RS(u_0)$, then the incoming flux is equal to the outgoing one, i.e.:

$$\sum_{i=1}^{n} f_i(\hat{u}_i) = \sum_{j=n+1}^{n+m} f_j(\hat{u}_j).$$

Such conditions prescribe exactly that the sum of traces of fluxes over incoming edges is equal to the sum of traces of fluxes over outgoing edges.

4.3 A Wave-Front Tracking Algorithm on Networks

Fix a network $(\mathcal{I}, \mathcal{J})$ and assume that the evolution on each edge is given by (4.2.1).
Once Riemann solvers at vertices are given, we can try to construct piecewise constant approximations via a wave-front tracking algorithm. The construction is very similar to that for systems of conservation law on a real line; see Section 2.6 of Chapter 2.

Let $u_0 = (u_{1,0}, \ldots, u_{N,0})$ be a piecewise constant map defined on the network. We begin by solving the Riemann Problems on each edge in correspondence of the jumps of $\bar{u}$ and the Riemann problems at vertices determined by the values of $\bar{u}$. We split rarefaction waves in rarefaction shock fans.

When a wave interacts with another one on an edge, we simply solve the new Riemann problem. Instead, when a wave reaches a vertex, we solve the Riemann problem at the vertex using the corresponding Riemann solver; see Figure 4.2.

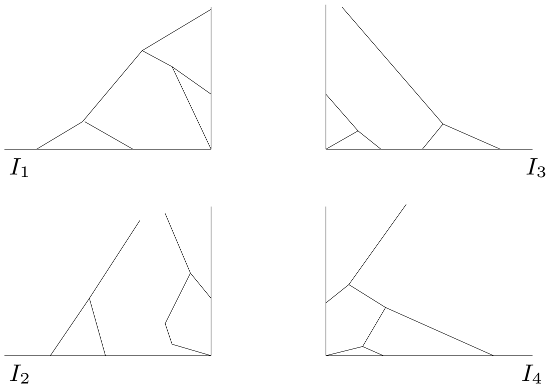


Fig. 4.2. Wave-front tracking in a network composed by a vertex J, two incoming edges I_1 and I_2 and two outgoing edges I_3 and I_4.

As for the wave-front tracking on a real line, it is fundamental to obtain three bounds.

1. Bound on the number of waves;
2. Bound on the number of interactions (of waves with other waves and vertices);
3. Bound on the total variation of the piecewise constant solution at every time in terms of the initial total variation.

4.3.1 The Scalar Case

Let us now restrict to the scalar case, i.e. $p = 1$, and assume:

(H) The functions u_i take values on a bounded set, say $[0,1]$, and RS takes values on $[0,1]^{n+m}$ for every vertex J with n incoming and m outgoing edges. The flux $f_i = f$ does not depend on the edge and is a strictly concave C^2 function with $f(0) = f(1) = 0$. Thus f has a unique point of maximum $\sigma \in]0,1[$.

Notice that such scalar system is in fact genuinely nonlinear (the linearly degenerate case being easier).

To obtain the estimates on number of waves and interactions, we need to introduce some notations and provide preliminary results.

Definition 4.3.1. *Let $\tau : [0,1] \to [0,1]$ be the map such that:*

1. *$f(\tau(u)) = f(u)$ for every $u \in [0,1]$;*
2. *$\tau(u) \neq u$ for every $u \in [0,1] \setminus \{\sigma\}$.*

Proposition 4.3.2. *The function τ is well defined and continuous. Moreover it satisfies*

$$0 \leqslant u \leqslant \sigma \Longleftrightarrow \sigma \leqslant \tau(u) \leqslant 1, \qquad \sigma \leqslant u \leqslant 1 \Longleftrightarrow 0 \leqslant \tau(u) \leqslant \sigma. \tag{4.3.2}$$

Proof. Fix $u \in [0,1]$. If $u = \sigma$, then $\tau(u) = u$, since there is just one point of maximum for f. If $u \neq \sigma$, then by 1., $\tau(u)$ can assume at most two values. One is u itself, while the other belongs to $]\sigma, 1]$ if $u < \sigma$ or it belongs to $[0, \sigma[$ if $u > \sigma$. Since we want that $\tau(u) \neq u$ if $u \neq \sigma$, then τ is clearly well defined and satisfies (4.3.2).

The continuity of τ follows from the regularity of the flux f. □

The following proposition describes the regions, which the images of all the possible Riemann solvers may belong to.

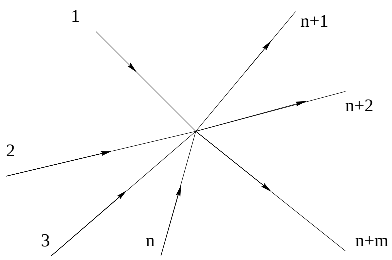


Fig. 4.3. a vertex with n incoming edges and m outgoing edges.

Proposition 4.3.3. *Fix a vertex J, an initial datum $(u_{1,0}, \ldots, u_{n+m,0})$ and a Riemann solver RS satisfying assumptions* **(Cons.1)** *and* **(Cons.2)**. *Define*

$$(\hat{u}_1, \ldots, \hat{u}_{n+m}) = RS(u_{1,0}, \ldots, u_{n+m,0}).$$

For an incoming edge I_i the following possibilities hold:

1. if the initial datum $u_{i,0} \in [0, \sigma]$*, then*

$$\hat{u}_i \in \{u_{i,0}\} \cup]\tau(u_{i,0}), 1];$$

2. if the initial datum $u_{i,0} \in [\sigma, 1]$*, then*

$$\hat{u}_i \in [\sigma, 1].$$

For an outgoing edge I_j *the following possibilities holds:*

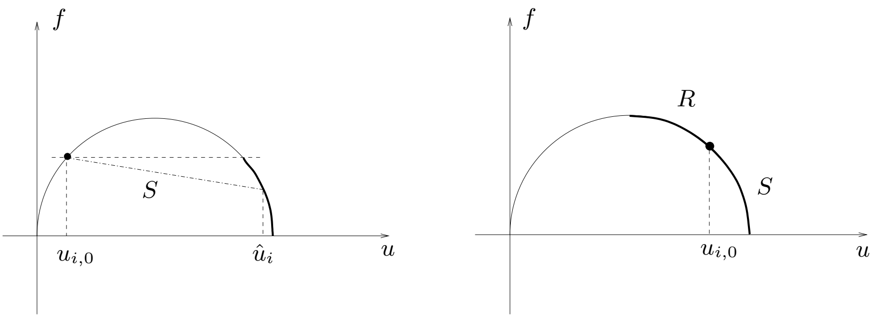
1. if the initial datum $u_{j,0} \in [0, \sigma]$*, then*

$$\hat{u}_j \in [0, \sigma];$$

2. if the initial datum $u_{j,0} \in [\sigma, 1]$*, then*

$$\hat{u}_j \in \{u_{j,0}\} \cup [0, \tau(u_{j,0})[.$$

Proof. Consider an incoming edge I_i. We require that the wave, produced by the solution to a Riemann problem with initial datum $(u_{i,0}, \hat{u}_i)$, has negative speed. So, if $u_{i,0} \in [0, \sigma]$, then $\hat{u}_i$ either is $u_{i,0}$ or belongs to $]\tau(u_{i,0}, 1]$. In the first case there is no wave, while in the second case the wave $(u_{i,0}, \hat{u}_i)$ is a shock wave with negative speed; see Figure 4.4 (a).



(a) First case. (b) Second case.

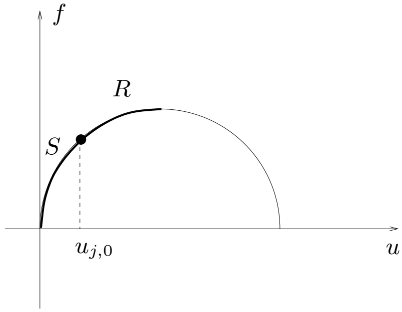
Fig. 4.4. Images of Riemann solvers in incoming edges.

If instead $u_{i,0} \in [\sigma, 1]$, then $\hat{u}_i$ belongs to $[\sigma, 1]$ and the wave $(u_{i,0}, \hat{u}_i)$ is a rarefaction or a shock wave with negative speed; see Figure 4.4 (b).

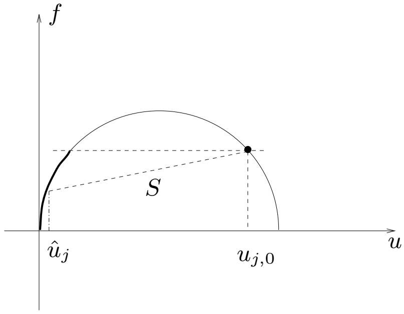
The case of an outgoing edge I_j is completely analogous; see Figures 4.5 (a) and (b). □

The previous proposition allows us to introduce the following functions. For each incoming edge I_i, define

$$\gamma_i^{max}(u_{i,0}) = \begin{cases} f(u_{i,0}), & \text{if } u_{i,0} \in [0, \sigma], \\ f(\sigma), & \text{if } u_{i,0} \in]\sigma, 1], \end{cases} \tag{4.3.3}$$



(a) First case.



(b) Second case.

Fig. 4.5. Images of Riemann solvers in outgoing edges.

while for each outgoing edge I_j, define

$$\gamma_j^{max}(u_{j,0}) = \begin{cases} f(\sigma), & \text{if } u_{j,0} \in [0, \sigma], \\ f(u_{j,0}), & \text{if } u_{j,0} \in]\sigma, 1]. \end{cases} \tag{4.3.4}$$

The quantities $\gamma_i^{max}(u_{i,0})$ and $\gamma_j^{max}(u_{j,0})$ represent the maximum flux that can be obtained by a single wave solution on each road.

From Proposition 4.3.3, we can deduce some properties of waves generated at a vertex. For this we need to introduce some notation.

Definition 4.3.4. *Fix an approximate wave front tracking solution u and a edge I_i, $i = 1, \ldots, N$. A wave θ in I_i is said a* big shock *if $u_-^\theta < u_+^\theta$ and*

$$sgn(u_-^\theta - \sigma) \cdot sgn(u_+^\theta - \sigma) < 0.$$

Definition 4.3.5. *Fix an approximate wave front tracking solution u and a vertex J. We say that an incoming edge I_i has a* good datum *at J at time $t > 0$ if*

$$u_i(t, b_i-) \in [\sigma, 1]$$

and a bad datum *otherwise. We say that an outgoing edge I_i has a* good datum *at J at time $t > 0$ if*

$$u_i(t, a_i+) \in [0, \sigma]$$

and a bad datum *otherwise.*

Lemma 4.3.6. *If an edge I_i of a vertex J has a good datum, then it remains good after interactions with J of waves coming from other edges. Moreover, no big shock can be produced in this way. If an edge I_i has a bad datum, then after any interaction with J of waves coming from other edges, either the datum of I_i is unchanged or a big shock is produced (and the new datum is good).*

Proof. Let us consider the case in which I_i is an incoming edge. By Proposition 4.3.3, if $u_{i,0}(t, b_i-) \in [\sigma, 1]$ (good datum), then the new state $\hat{u}_i \in [\sigma, 1]$ and no big shock is produced. If $u_i(t, b_i-) \in [0, \sigma[$ (bad datum), then the new state $\hat{u}_i \in \{u_i(t, b_i-)\} \cup]\tau(u_i(t, b_i-), 1]$, hence, if a wave is created, it is a big shock.

In the case of outgoing edge, if $u_i(t, a_i+) \in [0, \sigma]$ (good datum), then the new state $\hat{u}_i \in [0, \sigma]$, thus the resulting wave is not a big shock. Instead, if $u_i(t, a_i+) \in]\sigma, 1]$ (bad datum), then the new state $\hat{u}_i \in \{u_i(t, a_i+)\} \cup [0, \tau(u_i(t, a_i+))[$, therefore, if a wave is created, it is a big shock. □

Notice that, for $t > 0$, the number of waves may increase only for interactions of waves with vertices. When this happens some waves exiting the vertex are generated. If these waves do not come back to the vertex, then one easily get a bound on the number of waves and thus of interactions.
Hence, the main problem is to determine when a wave, generated at a vertex, may come back to the same vertex, after interactions with some waves inside an edge.

Lemma 4.3.7. *If a wave produced from a vertex J on an incoming edge I_i comes back to J, interacting only with waves produced by J, then the wave connects a bad left datum to a right good datum. The converse is true for outgoing edges.*

Proof. From Lemma 4.3.6, if there is a good datum on I_i at J only waves connecting good data are produced. If such waves interact on the edge I_i, then only waves with negative speed are produced.
Therefore the only wave which can come back to J is a big shock produced when I_i has a bad datum at J. By Lemma 4.3.6 we conclude. □

Since the speeds of waves are bounded, we can provide a general result on existence of wave-front tracking approximations. More precisely we have the following:

Proposition 4.3.8. *Consider a network $(\mathcal{I}, \mathcal{J})$, an evolution on each edge given by (4.2.1) with $p = 1$ and a Riemann solver RS_J, assigned for every $J \in \mathcal{J}$, satisfying assumptions* **(Cons.1)** *and* **(Cons.2)**. *Assume (H) holds true and let u_0 be piecewise constant (taking a finite number of values) initial datum. Then, for every $T > 0$, there exists a wave-front tracking approximate solution on $[0, T]$.*

Proof. Let $u(\cdot)$ be a wave front tracking approximate solution starting from u_0. If the number of waves remains bounded and the same happens for the number of interactions, then u is well defined on $[0, T]$.

Consider the set A of all indices i such that $b_i - a_i < +\infty$ and set:

$$\delta = \frac{\min_{i \in A}\{b_i - a_i\}}{\max\{|f'(w)| : w \in [0,1]\}},$$

then δ is the minimum time for a wave to go from a vertex to another one. Hence if we can bound the number of waves and interactions locally at each vertex J, then a global bound follows from $\delta > 0$.

Let us now focus on a fixed vertex J. If the number of interactions with J of waves from any edge is bounded, then, the number of waves is also automatically bounded. Hence the only problem for definition of u is when there is an infinite number of interactions with J accumulating at some time $\bar{t} \leq T$. If this happens, then, by Lemma 4.3.7, there are big shocks bouncing back and forth from J.

We want to prove that the solution u can be defined up to time $\bar{t}$ via a limiting procedure. Notice that the number of waves on each edge I_i is bounded up to time $\bar{t}$. On each edge two possibility occurs:

c1. the interaction times of big shocks with J from the edge accumulate at $\bar{t}$;
c2. the interaction times of big shocks with J from the edge do not accumulate at $\bar{t}$.

In case c1., for times in a left neighborhood of $\bar{t}$, all waves produced from J, except big shocks, interact with a big shock and are cancelled. Therefore the limit $\lim_{t \to \bar{t}-} u(t)$ is well defined and contain no wave in a neighborhood of the vertex.

In case c2. it may happen that an infinite number of waves are produced for times in a left neighborhood of $\bar{t}$ from the vertex J. However, the datum on the edge is good in a left neighborhood of $\bar{t}$. Then no wave comes back to the vertex and again we can define the limit $\lim_{t \to \bar{t}-} u(t)$. In fact, due to the genuinely nonlinearity, there is a cancellation of waves for every $t > \bar{t}$, thus $u(t)$ is in BV for t in a right neighborhood of $\bar{t}$; see [50].

We conclude that $u(\bar{t})$ can be defined by a limiting procedure. Since the limiting procedure is always possible, we conclude that the solution can be defined up to time T by a transfinite induction. □

The above procedure is highly non constructive, thus it is not possible to pass to the limit with wave front tracking approximations. Hence we give some conditions ensuring a constructive method for generating wave front tracking approximate solutions.

Theorem 4.3.9. *Consider a network $(\mathcal{I}, \mathcal{J})$, an evolution on each edge given by (4.2.1) with $p = 1$ and a Riemann solver RS_J, assigned for every $J \in$*

$\mathcal{J}$, satisfying assumptions **(Cons.1)** *and* **(Cons.2)**. *Let u_0 be a piecewise constant (taking a finite number of values) initial datum, assume (H) and:*

(H^*) For every vertex J, consider a network formed by only the vertex J, replacing incoming and outgoing edges by infinite length ones. Then there exists a constant C_J, such that, using the corresponding Riemann solver RS_J the following holds. For every wave front tracking approximate solution, denoting by M the number of waves in the initial datum, at most $C_J M$ waves are produced by the vertex J and there are at most $C_J M$ interactions of waves with the vertex J.

Then, for every $T > 0$, we can construct a wave front tracking approximate solution on $[0, T]$.

Proof. Define δ as in the proof of Proposition 4.3.8, then δ is the minimum time for a wave to go from a vertex to another one.
Denote by $u(\cdot)$ the wave-front tracking approximation with initial datum u_0. Define M_i^k to be the number of waves of $u(k\delta)$ on edge I_i and M_J^k to be the number of waves produced by J on the interval $[k\delta, (k+1)\delta]$. By assumption (H^*), for every junction J we have:

$$M_J^k \leq C_J \max_{\ell} M_\ell^k,$$

where the maximum is taken over all edges I_ℓ incident at J, and for every edge I_i connecting vertex J_1 with vertex J_2 we have:

$$M_i^{k+1} \leq M_i^k + \max\{M_{J_1}^k, M_{J_2}^k\}.$$

Setting:

$$C_{\mathcal{J}} = \max_{J \in \mathcal{J}} C_J, \qquad M_{\mathcal{I}}^k = \max_{i=1,\dots,N} M_i^k, \qquad M_{\mathcal{J}}^k = \max_{J \in \mathcal{J}} M_J^k,$$

we get:

$$M_{\mathcal{J}}^k \leq C_{\mathcal{J}} M_{\mathcal{I}}^k, \qquad M_{\mathcal{I}}^{k+1} \leq M_{\mathcal{I}}^k + M_{\mathcal{J}}^k.$$

By recursion:

$$M_{\mathcal{I}}^k \leq (1 + C_{\mathcal{J}})^k M_{\mathcal{I}}^0,$$

thus proving that the number of waves is bounded. Similarly we can bound the number of interactions, thus concluding. □

4.3.2 Rich Systems

For the case $p > 1$, we can provide some bounds only for special classes of systems. Following [98], we define:

Definition 4.3.10. *A hyperbolic system of conservation laws (4.2.1) is called rich on a region $D \subset \mathbb{R}^p$ if the following holds.*

- *The region D is invariant for solutions to Riemann problems. That is for every Riemann data $u_-, u_+ \in D$ there exists a solution to the Riemann problem which takes values in D.*
- *Each characteristic family either is linearly degenerate or genuinely nonlinear with coinciding shocks and rarefactions curves.*
- *There exist disjoint closed intervals $\Lambda_i \subset \mathbb{R}$, such that every wave of the i–th family connecting data in D has velocity in Λ_i.*
- *The intervals Λ_i do not contain the value zero.*

Theorem 4.3.11. *Consider a network $(\mathcal{I}, \mathcal{J})$, an evolution on each edge given by a rich system (4.2.1) and a Riemann solver RS_J assigned for every $J \in \mathcal{J}$. Let u_0 be piecewise constant (taking a finite number of values) initial datum and assume $u_0(\cdot) \in D$.*

Then, for every $T > 0$, the number of waves and the number of interactions are bounded for a wave-front tracking solution from u_0 on $[0, T]$.

Proof. Notice that for a rich system the following holds:

- if two waves of the i–th family interact on an edge, then only a wave of the i–th family is produced. Moreover, if one of the two waves is a rarefaction, then the resulting wave may be a rarefaction only of smaller size;
- if a wave of the i–th family interacts with a wave of the j–th family ($j \neq i$) on an edge, then the resulting waves are again only one of the i–th family and one of the j–th family.
- if a wave of the i–th family is produced by an edge, then it can not come back to the same edge by interactions with other waves.

Therefore, in the case of a rich system, it is clear that an assumption similar to (H^*) holds. Moreover, as for the scalar case, there exists lower bound $\delta > 0$ for the time taken by a wave to go from a vertex to another one. Then clearly the number of interactions and of waves is bounded on $[0, T]$. □

4.4 A Case Study for Riemann Solvers

Let us consider a network composed by two edges I_1 and I_2 connected together by a vertex J. I_1 is the incoming edge, modeled by the interval $]-\infty, 0]$, while I_2 is the outgoing one, modeled by the interval $[0, +\infty[$.

In the incoming edge I_1, the evolution is described by the conservation law

$$u_t(t, x) + g(u(t, x))_x = 0, \tag{4.4.5}$$

where $u(t, x) \in [u_a^i, u_b^i]$ and g is the corresponding flux. On the outgoing edge I_2, the evolution is described by the conservation law

$$u_t(t, x) + f(u(t, x))_x = 0, \tag{4.4.6}$$

where $u \in [u_a^o, u_b^o]$ and f is the flux. We assume that the fluxes $f : [u_a^o, u_b^o] \to \mathbb{R}$ and $g : [u_a^i, u_b^i] \to \mathbb{R}$ satisfy the following properties:

1. f and g are strictly concave functions;
2. there exists $\sigma_g \in]u_a^i, u_b^i[$ such that $g(\sigma_g) \geqslant g(u)$ for every $u \in [u_a^i, u_b^i]$;
3. there exists $\sigma_f \in]u_a^o, u_b^o[$ such that $f(\sigma_f) \geqslant f(u)$ for every $u \in [u_a^o, u_b^o]$.

Define

$$\gamma_a^i := g(u_a^i),\ \gamma_b^i := g(u_b^i),\ \gamma_a^o := f(u_a^o),\ \gamma_b^o := f(u_b^o),$$

see Figure 4.6.

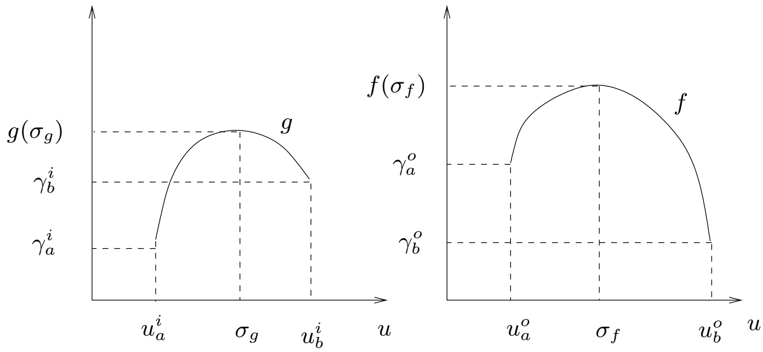


Fig. 4.6. Graphs of the fluxes f and g.

Consider the Riemann problem at J

$$\begin{cases} u_t + g(u)_x = 0, & \text{if } x < 0,\, t > 0, \\ u_t + f(u)_x = 0, & \text{if } x > 0,\, t > 0, \\ u(0,x) = u_l, & \text{if } x < 0, \\ u(0,x) = u_r, & \text{if } x > 0, \end{cases} \tag{4.4.7}$$

where $u_l \in [u_a^i, u_b^i]$ and $u_r \in [u_a^o, u_b^o]$.

We want to describe Riemann solvers at J satisfying some "physical" conditions as the conservation of the quantity u at J. Again we express the solution by assigning the boundary values.

Definition 4.4.1. *We say that $u^- \in [u_a^i, u_b^i]$ and $u^+ \in [u_a^o, u_b^o]$ determine a weak solution to the Riemann problem (4.4.7) at J if*

(R-1) the wave (u_l, u^-) on I_1 has negative speed;
(R-2) the wave (u^+, u_r) on I_2 has positive speed;
(R-3) $g(u^-) = f(u^+)$.

The weak solution to the Riemann problem at J is given by the waves (u_l, u^-) and (u^+, u_r) respectively on I_1 and I_2.

We look for conditions on γ_a^i, γ_b^i, γ_a^o, γ_b^o in order that, for every $u_l \in [u_a^i, u_b^i]$ and for every $u_r \in [u_a^o, u_b^o]$ the Riemann problem (4.4.7) admits at least a weak solution satisfying (R-1), (R-2) and (R-3). The following lemmas hold.

Lemma 4.4.2. *Assume $\gamma_a^i \leqslant \gamma_b^i$, $\gamma_a^o \geqslant \gamma_b^o$. The Riemann problem (4.4.7) admits a weak solution satisfying (R-1), (R-2) and (R-3) for every initial condition if and only if $\gamma_a^i = \gamma_b^i = \gamma_a^o = \gamma_b^o$.*

Proof. If $\gamma_a^i = \gamma_b^i = \gamma_a^o = \gamma_b^o$, then

$$u^- = \begin{cases} u_b^i, & \text{if } u_l \neq u_a^i, \\ u_a^i, & \text{if } u_l = u_a^i, \end{cases}$$

and

$$u^+ = \begin{cases} u_a^o, & \text{if } u_r \neq u_b^o, \\ u_b^o, & \text{if } u_r = u_b^o, \end{cases}$$

provide a weak solution to the Riemann problem satisfying (R-1), (R-2) and (R-3).

Suppose now that the Riemann problem (4.4.7) admits a least one weak solution satisfying (R-1), (R-2) and (R-3) for every initial condition.

Assume by contradiction that $\gamma_a^i < \gamma_b^i$. Fix u_l such that $g(u_l) < \gamma_b^i$. By (R-1), $u^- = u_l$ and so, by (R-3), $f(u^+) = g(u_l)$. This implies that $\gamma_a^o \leqslant g(u_l)$ and $\gamma_b^o \geqslant g(u_l)$, otherwise, if $\gamma_a^o > g(u_l)$, then the Riemann problem with initial condition $(u_l, u_r) = (u_l, u_a^o)$ does not admit weak solutions, while, if $\gamma_b^o < g(u_l)$, then the Riemann problem with initial condition $(u_l, u_r) = (u_l, u_b^o)$ does not admit weak solutions. Thus we have $g(u_l) = \gamma_a^0 = \gamma_b^o$, that is a contradiction since the arbitrariness of u_l. Therefore $\gamma_a^i = \gamma_b^i$.

By contradiction assume that $\gamma_a^o > \gamma_b^o$. Fixing u_r such that $f(u_r) < \gamma_a^o$ as in the previous case, we conclude that $f(u_r) = \gamma_a^i = \gamma_b^i$, that is a contradiction. So $\gamma_a^o = \gamma_b^o$.

Taking now $(u_l, u_r) = (u_a^i, u_b^o)$, we conclude that $\gamma_a^i = \gamma_b^i = \gamma_a^o = \gamma_b^o$; see Figure 4.7. □

Lemma 4.4.3. *Assume $\gamma_a^i > \gamma_b^i$, $\gamma_a^o \geqslant \gamma_b^o$. The Riemann problem (4.4.7) admits a weak solution satisfying (R-1), (R-2) and (R-3) for every initial condition if and only if $\gamma_b^i \leqslant \gamma_b^o \leqslant \gamma_a^o \leqslant \gamma_a^i$.*

Proof. Consider first the case $\gamma_b^i \leqslant \gamma_b^o \leqslant \gamma_a^o \leqslant \gamma_a^i$. For every u_l define the set

$$A^-(u_l) := \left\{ \tilde{u} \in [u_a^i, u_b^i] : \text{ the wave } (u_l, \tilde{u}) \text{ has negative speed} \right\}.$$

We have that

$$[\gamma_b^i, \gamma_a^i] \subseteq g(A^-(u_l))$$

for every u_l. If u_r satisfies $f(u_r) < \gamma_a^o$, then $u^+ = u_l$ and there exists an element in $A^-(u_l)$ satisfying (R-3). If instead u_r satisfies $f(u_r) \geqslant \gamma_a^o$, then

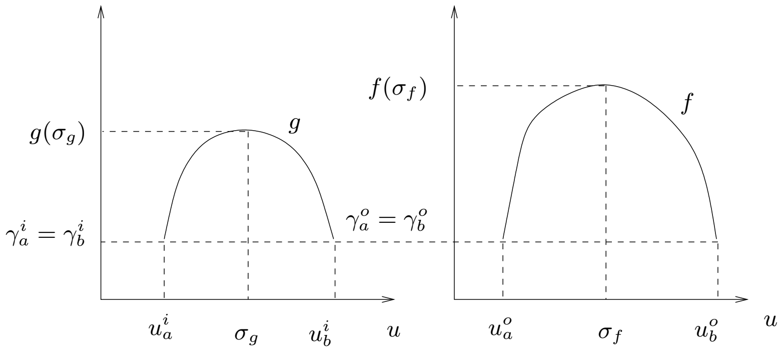


Fig. 4.7. The fluxes f and g in the case of Lemma 4.4.2

there exists a weak solution such that $u^+ = u_a^o$. Thus the sufficient condition is proved.

Assume now that the Riemann problem (4.4.7) admits a weak solution satisfying (R-1), (R-2) and (R-3) for every initial condition.

Suppose first by contradiction that $\gamma_b^i > \gamma_b^o$. Consider $u_r = u_b^o$. Then, by (R-2), $u^+ = u_l$ and so it is not possible to satisfy (R-3). Therefore $\gamma_b^i \leqslant \gamma_b^o$.

Suppose now that $\gamma_a^o > \gamma_a^i$. Consider $(u_l, u_r) = (u_a^i, u_a^o)$. By (R-1), u^- satisfies $g(u^-) \leqslant \gamma_a^i$. By (R-2), u^+ satisfies $f(u^+) \geqslant \gamma_a^o$. Then (R-3) is not satisfied and so we get $\gamma_a^o \leqslant \gamma_a^i$; see Figure 4.8.

This concludes the lemma. □

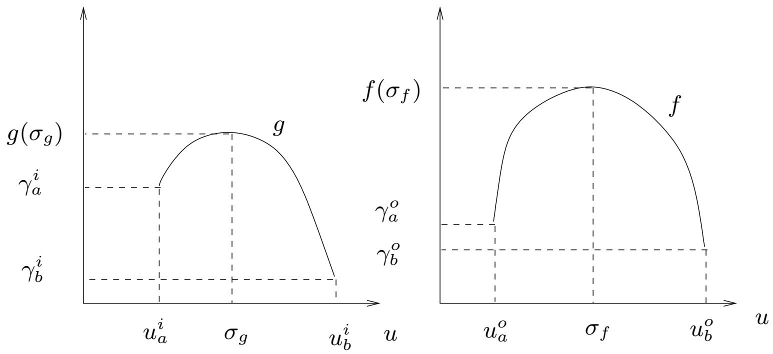


Fig. 4.8. The fluxes f and g in the case of Lemma 4.4.3

Lemma 4.4.4. *Assume $\gamma_a^i \leqslant \gamma_b^i$, $\gamma_a^o < \gamma_b^o$. The Riemann problem (4.4.7) admits a weak solution satisfying (R-1), (R-2) and (R-3) for every initial condition if and only if $\gamma_a^o \leqslant \gamma_a^i \leqslant \gamma_b^i \leqslant \gamma_b^o$.*

Proof. The proof is given in the same way as in Lemma 4.4.3, since the situation is completely symmetric. □

Lemma 4.4.5. *Assume $\gamma_a^i > \gamma_b^i$, $\gamma_a^o < \gamma_b^o$. The Riemann problem (4.4.7) admits a weak solution satisfying (R-1), (R-2) and (R-3) for every initial condition if and only if $\gamma_a^o \leqslant \gamma_a^i$ and $\gamma_b^i \leqslant \gamma_b^o$.*

Proof. Assume first that $\gamma_a^o \leqslant \gamma_a^i$ and $\gamma_b^i \leqslant \gamma_b^o$. For every u_l and u_r define the sets

$$A^-(u_l) := \left\{\tilde{u} \in [u_a^i, u_b^i] : \text{ the wave } (u_l, \tilde{u}) \text{ has negative speed}\right\}$$

and

$$A^+(u_r) := \{\tilde{u} \in [u_a^o, u_b^o] : \text{ the wave } (\tilde{u}, u_r) \text{ has positive speed}\} .$$

We have that

$$[\gamma_b^i, \gamma_a^i] \subseteq g(A^-(u_l)), \quad [\gamma_a^o, \gamma_b^o] \subseteq f(A^+(u_r)),$$

for every u_l and u_r. By assumption

$$[\gamma_b^i, \gamma_a^i] \cap [\gamma_a^o, \gamma_b^o] \neq \emptyset$$

and so it is possible to find $u^- \in A^-(u_l)$ and $u^+ \in A^+(u_r)$ such that $f(u^+) = g(u^-)$. Hence the sufficient condition is proved.

Assume now that the Riemann problem (4.4.7) admits a weak solution satisfying (R-1), (R-2) and (R-3) for every initial condition.

Suppose by contradiction that $\gamma_a^i < \gamma_a^o$. If $u_l = u_a^i$, then u^- by (R-1) satisfies $g(u^-) \leqslant \gamma_a^i$ and so (R-3) can not be satisfied. Therefore $\gamma_a^i \geqslant \gamma_a^o$.

Suppose now that $\gamma_b^i > \gamma_b^o$. If $u_r = u_b^o$, then u^+ satisfies $f(u^+) \leqslant \gamma_b^o$ and so (R-3) can not be satisfied. Thus $\gamma_b^i \leqslant \gamma_b^o$ (see Figure 4.9) and the proof is finished. □

The situation is summarized in Table 4.1.

	Situation	Conditions
1	$\gamma_a^i \leqslant \gamma_b^i$ and $\gamma_a^o \geqslant \gamma_b^o$	$\gamma_a^i = \gamma_b^i = \gamma_a^o = \gamma_b^o$ (Lemma 4.4.2)
2	$\gamma_a^i > \gamma_b^i$ and $\gamma_a^o \geqslant \gamma_b^o$	$\gamma_b^i \leqslant \gamma_b^o \leqslant \gamma_a^o \leqslant \gamma_a^i$ (Lemma 4.4.3)
3	$\gamma_a^i \leqslant \gamma_b^i$ and $\gamma_a^o < \gamma_b^o$	$\gamma_a^o \leqslant \gamma_a^i \leqslant \gamma_b^i \leqslant \gamma_b^o$ (Lemma 4.4.4)
4	$\gamma_a^i > \gamma_b^i$ and $\gamma_a^o < \gamma_b^o$	$\gamma_a^o \leqslant \gamma_a^i$ and $\gamma_b^i \leqslant \gamma_b^o$ (Lemma 4.4.5)

Table 4.1. Conditions in order the Riemann problem at junction admits at least one weak solution satisfying (R-1), (R-2) and (R-3).

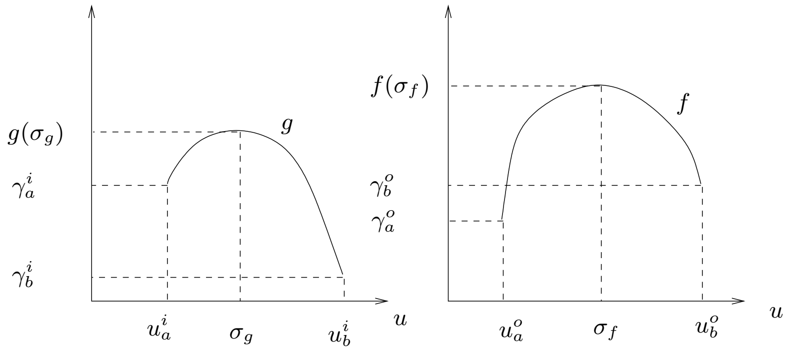


Fig. 4.9. The fluxes f and g in the case of Lemma 4.4.5

4.4.1 Construction of Riemann Solvers

Consider the Riemann problem at J (4.4.7). We restrict ourselves to the case $u_a^i = u_a^o = 0$, $u_b^i = u_b^o = 1$ and $\gamma_a^i = \gamma_b^i = \gamma_a^o = \gamma_b^o = 0$.

We introduce a definition of admissible Riemann solver which takes into account conservation of the quantity u (thus also to have the traces taking the boundary values) and some continuity conditions.

Definition 4.4.6. *A Riemann solver for the Riemann problem (4.4.7)*

$$RS(u_l, u_r) = (RS_1(u_l, u_r), RS_2(u_l, u_r)) = (u^-, u^+)$$

is admissible if

(H1). $g(u^-) = f(u^+)$*;*
(H2). the wave (u_l, u^-) *has negative speed, while the wave* (u^+, u_r) *has positive speed;*
(H3). the function $(u_l, u_r) \mapsto (g(u^-), f(u^+))$ *is continuous;*
(H4). for every $\tilde{u}$ *such that the wave* $(\tilde{u}, RS_1(u_l, u_r))$ *has positive speed the following holds:*

$$g(RS_1(\tilde{u}, u_r)) \in [\min\{g(RS_1(u_l, u_r)), g(\tilde{u})\}, \max\{g(RS_1(u_l, u_r)), g(\tilde{u})\}]; \tag{4.4.8}$$

(H5). for every $\tilde{u}$ *such that the wave* $(RS_2(u_l, u_r), \tilde{u})$ *has negative speed the following holds:*

$$f(RS_2(u_l, \tilde{u})) \in [\min\{f((RS_2(u_l, u_r)), f(\tilde{u})\}, \max\{f((RS_2(u_l, u_r)), f(\tilde{u})\}]. \tag{4.4.9}$$

Definition 4.4.7. *A couple* (u_l, u_r) *is said an equilibrium if* $RS(u_l, u_r) = (u_l, u_r)$*.*

Let us now describe all admissible Riemann solvers for (4.4.7). We treat only the case $f(\sigma_f) \geqslant g(\sigma_g)$, the others being similar.
There are some different possibilities:
1. $u_l \in [\sigma_g, 1]$ and $u_r \in [0, \sigma_f]$; see Figure 4.10. Since the waves produced must have negative speed in the incoming edge and positive speed in the outgoing edge, then $u^- \in [\sigma_g, 1]$ and $u^+ \in [0, \sigma_f]$. By hypothesis (H3), there exists a continuous function

$$\Gamma : [0, g(\sigma_g)] \times [0, f(\sigma_f)] \to [0, g(\sigma_g)] \tag{4.4.10}$$

such that

$$g(u^-) = f(u^+) = \Gamma(g(u_l), f(u_r)).$$

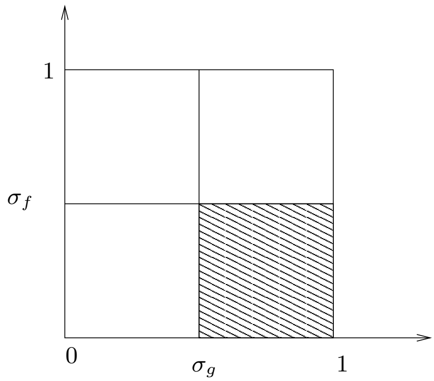


Fig. 4.10. The region considered in case 1.

By (CC) of Definition 4.2.2 we deduce that, if $a \in \operatorname{Im} \Gamma$, then $\Gamma(a, a) = a$ and so, every element of the image of Γ is the flux of an equilibrium for the Riemann problem. Conversely, if (u_l, u_r) is an equilibrium for the Riemann problem, then

$$\Gamma(g(u_l), f(u_r)) = f(u_r) = g(u_l),$$

and so the image of Γ coincides with the set X defined by

$$X := \{s \in [0, g(\sigma_g)] : \exists (u_l, u_r) \in [\sigma_g, 1] \times [0, \sigma_f] \text{ equilibrium}, \ g(u_l) = f(u_r) = s\}. \tag{4.4.11}$$

We have the following characterization of the set X.

Lemma 4.4.8. *X is a closed, non empty and connected set. Thus $X = [\bar{\gamma}_1, \bar{\gamma}_2]$, with $0 \leqslant \bar{\gamma}_1 \leqslant \bar{\gamma}_2 \leqslant g(\sigma_g)$.*

Proof. X is a connected set since it is the image of a connected set through a continuous function. Moreover X is clearly non empty. Finally we take $x \in \bar{X}$ and a sequence $a_n \to x$ such that $a_n \in X$ for every $n \in \mathbb{N}$. We have:

$$\Gamma(x,x) = \lim_{n\to+\infty} \Gamma(a_n,a_n) = \lim_{n\to+\infty} a_n = x$$

and so $x \in X$. □

From now on with $\bar{\gamma}_1$ and $\bar{\gamma}_2$ we denote respectively the minimum and maximum of the set X.

2. $u_l \in [0,\sigma_g[$ and $u_r \in [0,\sigma_f]$; see Figure 4.11. Some different cases are possible.

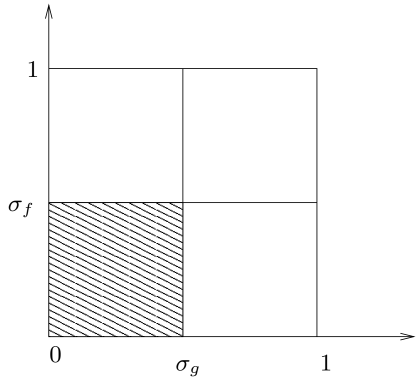


Fig. 4.11. The region considered in case 2.

a. $g(u_l) \leqslant \bar{\gamma}_1$. By (H2), u^- either is u_l or belongs to $[\sigma_g,1]$ and $g(u^-) < g(u_l)$. The second possibility can not happen since otherwise $g(u^-) < \bar{\gamma}_1$, a contradiction with (CC) of Definition 4.2.2, and so the solution is given by (u_l,u^+) with $u^+ \in [0,\sigma_f[$, $f(u^+) = g(u_l)$.

b. $g(u_l) > \bar{\gamma}_2$. We claim that $u^- \in [\sigma_g,1]$ and $g(u^-) \in X$. Indeed, consider the function
$$\begin{aligned} h_{u_r} : [0,\sigma_g] &\to [0,g(\sigma_g)] \\ u_l &\mapsto g(u^-) \end{aligned}$$
giving the left flux of the solution to the Riemann problem with (u_l,u_r) initial states. It is continuous by (H3). Therefore
$$\lim_{r\to\sigma_g^-} h_{u_r}(r) = h_{u_r}(\sigma_g) \leqslant \bar{\gamma}_2,$$
by the analysis of possibility 1. Thus there exists a small left neighborhood V of σ_g such that $h_{u_r}(r) \leqslant \bar{\gamma}_2$ for every $r \in V$, otherwise, by (H2), there exists a sequence $r_n \to \sigma_g^-$ so that $h_{u_r}(r_n) = g(r_n) \geqslant g(r_0) > \bar{\gamma}_2$ contradicting the continuity of h_{u_r}. Consider now the set
$$Y := \{r \in [0,\sigma_g[: g(r) > \bar{\gamma}_2, h_{u_r}(r) > \bar{\gamma}_2\}$$

and suppose that $Y \neq \emptyset$. We define $\eta := \sup Y$. The previous analysis shows that

$$0 < \eta < \sigma_g, \qquad \bar{\gamma}_2 < g(\eta)$$

and by continuity of h_{u_r}

$$h_{u_r}(\eta) \geqslant g(\eta) > \bar{\gamma}_2.$$

Moreover

$$\lim_{r \to \eta^+} h_{u_r}(r) \leqslant \bar{\gamma}_2,$$

a contradiction. Thus $Y = \emptyset$ and the claim is proved.

c. $\bar{\gamma}_1 < g(u_l) \leqslant \bar{\gamma}_2$. In this case $h_{u_r}(u_l) \in [\bar{\gamma}_1, \bar{\gamma}_2]$. If $h_{u_r}(u_l) = g(u_l)$, then the solution is given by (u_l, u^+), where $u^+ \in [0, \sigma_f[$ with $f(u^+) = g(u_l)$. Otherwise, if $h_{u_r}(u_l) < g(u_l)$, then $u^- \in [\sigma_g, 1]$.

Remark 4.4.9. If $\tilde{\gamma} \in]\bar{\gamma}_1, \bar{\gamma}_2[$ satisfies $h_{u_r}(u_l) = \tilde{\gamma}$ for $(u_l, u_r) \in [0, \sigma_g[\times [0, \sigma_f]$ with $g(u_l) = f(u_r) = \tilde{\gamma}$, then condition (H4) implies that

$$h_{u_r}(r) = g(r)$$

for every $r \in [0, \sigma_g[$ such that $\bar{\gamma}_1 \leqslant g(r) \leqslant \tilde{\gamma}$.

3. $u_l \in [\sigma_g, 1]$ and $u_r \in]\sigma_f, 1]$; see Figure 4.12. This case is completely symmetric with respect to the previous one.

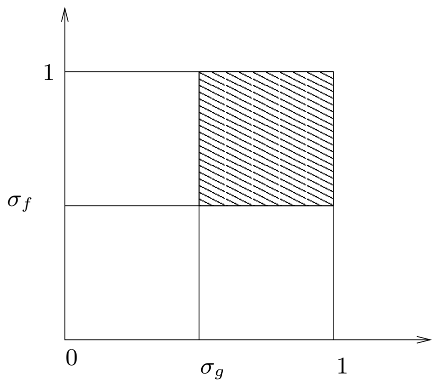


Fig. 4.12. The region considered in case 3.

4. $u_l \in [0, \sigma_g[$ and $u_r \in]\sigma_f, 1]$; see Figure 4.13. We have some different cases.

a. $\min\{g(u_l), f(u_r)\} \leqslant \bar{\gamma}_1$. Without loss of generalities we suppose that $g(u_l) \leqslant f(u_r)$. By (H2), u^- either is u_l or $u^- \in]\sigma_g, 1]$ with $g(u^-) < g(u_l)$.

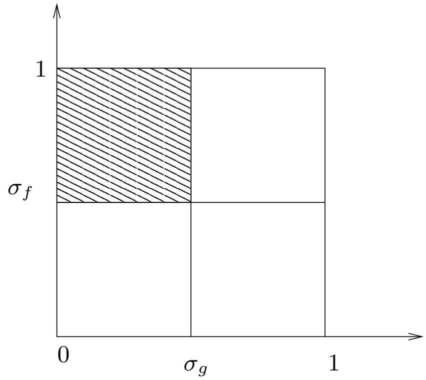


Fig. 4.13. The region considered in case 4.

Analogous u^+ either is u_r or $u^+ \in [0, \sigma_f[$ with $f(u^+) < f(u_r)$. If $u^- \in]\sigma_g, 1]$, then, by (H1), $u^+ \in [0, \sigma_f[$, but this is not an equilibrium. Thus $u^- = u_l$. If $f(u_r) = g(u_l)$, then $u^+ = u_r$ and the solution is (u_l, u_r). Otherwise if $f(u_r) > g(u_l)$, then $u^+ \in [0, \sigma_f[$, $f(u^+) = g(u_l)$ and the solution is (u_l, u^+).

b. $\bar{\gamma}_1 < \min\{g(u_l), f(u_r)\} \leqslant \bar{\gamma}_2$. Without loss of generalities we suppose $g(u_l) \leqslant f(u_r)$. If $g(u_l) < f(u_r)$, then $u^+ \in [0, \sigma_f[$ and the case is completely identical to 2.c.
   If $g(u_l) = f(u_r)$, then by (H3) the solution is uniquely determined by the previous case.
c. $\min\{g(u_l), f(u_r)\} > \bar{\gamma}_2$. Without loss of generalities we suppose that $g(u_l) \leqslant f(u_r)$. If $g(u_l) < f(u_r)$, then $u^+ \in [0, \sigma_f[$ by (H2) and also $u^- \in]\sigma_g, 1]$ by 2.b.
   If $g(u_l) = f(u_r)$, then by (H3) the solution is uniquely determined by the previous case.

Given an admissible Riemann solver at the vertex J, it is possible to define an admissible weak solution to (4.4.5) and (4.4.6).

Definition 4.4.10. *Fix an admissible Riemann solver RS. Let $u = (u_1, u_2)$ be such that u_1 and u_2 are of bounded variation for every $t \geqslant 0$. Then u is an admissible weak solution to (4.4.5) and (4.4.6) if*

1. *u_1 is a weak entropic solution to (4.4.5);*
2. *u_2 is a weak entropic solution to (4.4.6);*
3. *for almost every $t \geqslant 0$ the couple $(u_1(t, 0-), u_2(t, 0+))$ is an equilibrium for the Riemann solver RS.*

4.4.2 Case of X Singleton

This subsection is focused on the special case $X = \{\bar{\gamma}\}$. First we prove the following:

Theorem 4.4.11. *Consider an admissible Riemann Solver RS. Then for every Riemann data* (u_l, u_r) *there exists a unique centered weak entropic solution if and only if* X *is a singleton.*

Proof. Assume that there exist $s_1 \neq s_2$ such that both s_i belong to X. Consider some Riemann data (u_l^i, u_r^i) such that $g(u_l^i) = f(u_r^i) = s_i$, $u_l^i \in [\sigma_g, 1]$ and $u_r^i \in [0, \sigma_f]$. The Riemann problem with initial data (u_l^1, u_r^1) admits a centered weak solution formed by the wave (u_l^1, u_l^2) on the incoming edge and by the wave (u_r^2, u_r^1) on the outgoing edge. Since there is also the constant solution, uniqueness is violated.

If, on the contrary, X is a singleton, uniqueness is obtained by assumptions (H1)–(H5). □

In this case, the Riemann solver is completely described by the following possibilities.

1. $u_l \in [\sigma_g, 1]$ and $u_r \in [0, \sigma_f]$. In this case the solution to the Riemann problem satisfies $u^- \in [\sigma_g, 1]$, $u^+ \in [0, \sigma_f]$ and $g(u^-) = f(u^+) = \bar{\gamma}$.
2. $u_l \in [0, \sigma_g[$ and $u_r \in [0, \sigma_f]$. If $g(u_l) > \bar{\gamma}$, then the solution to the Riemann problem satisfies $u^- \in [\sigma_g, 1]$, $u^+ \in [0, \sigma_f]$ and $g(u^-) = f(u^+) = \bar{\gamma}$.
   If $g(u_l) \leqslant \bar{\gamma}$, then the solution to the Riemann problem satisfies $u^- = u_l$, $u^+ \in [0, \sigma_f]$ and $g(u^-) = f(u^+)$.
3. $u_l \in [\sigma_g, 1]$ and $u_r \in]\sigma_f, 1]$. The situation is completely symmetric to the previous case.
4. $u_l \in [0, \sigma_g[$ and $u_r \in]\sigma_f, 1]$. If $\min\{g(u_l), f(u_r)\} > \bar{\gamma}$, then the solution to the Riemann problem satisfies $u^- \in [\sigma_g, 1]$, $u^+ \in [0, \sigma_f]$ and $g(u^-) = f(u^+) = \bar{\gamma}$.
   If $\min\{g(u_l), f(u_r)\} \leqslant \bar{\gamma}$ and $g(u_l) = f(u_r)$, then the solution to the Riemann problem is (u_l, u_r).
   If $\min\{g(u_l), f(u_r)\} \leqslant \bar{\gamma}$ and $g(u_l) < f(u_r)$, then the solution to the Riemann problem satisfies $u^- = u_l$, $u^+ \in [\sigma_f, 1]$ and $g(u_l) = f(u^+)$.
   If $\min\{g(u_l), f(u_r)\} \leqslant \bar{\gamma}$ and $g(u_l) > f(u_r)$, then the solution to the Riemann problem satisfies $u^- \in [\sigma_g, 1]$, $u^+ = u_r$ and $g(u^-) = f(u_r)$.

Remark 4.4.12. If $f(\sigma_f) = g(\sigma_g)$ and $X = \{g(\sigma_g)\}$, then the Riemann solver is completely identical to that used in [27] and [1].

Remark 4.4.13. If there exists a unique $u^* \in]0, 1[$ such that $f(u^*) = g(u^*)$ and if $X = \{f(u^*)\}$, then the Riemann solver is identical to that used in [69].

4.4.3 Estimates of Flux Variation

In this section we estimate the total variation of the flux along an approximate wave-front tracking solution. This is a key step to construct weak solution for any initial data as shown in next Chapters.

Fix an approximate wave-front tracking solution u. The following lemma shows that the total variation of the flux remains constant when a wave interacts with J. This is due in particular by the properties (H4) and (H5) of the definition of admissible Riemann solver.

Lemma 4.4.14. *Consider the network composed by the incoming road I_1, the outgoing one I_2 and by the junction J. Fix an admissible Riemann solver RS at J. If a wave interacts with J at time $\bar{t}$, then*

$$\textit{Tot.Var.}\ [f(u(\bar{t}+,\cdot)) + g(u(\bar{t}+,\cdot))] = \textit{Tot.Var.}\ [f(u(\bar{t}-,\cdot)) + g(u(\bar{t}-,\cdot))]\,. \tag{4.4.12}$$

Proof. Fix an equilibrium (u_l, u_r). First suppose that a wave $(\tilde{u}, u_l)$ with positive speed interacts with J from the incoming edge I_1. We denote with (u^-, u^+) the solution to the Riemann problem at J with the initial datum $(\tilde{u}, u_r)$. We have

$$\begin{aligned}
\text{Tot.Var.}\ [f(u(\bar{t}+,\cdot)) + g(u(\bar{t}+,\cdot))] &= \left|g(\tilde{u}) - g(u^-)\right| + \left|f(u^+) - f(u_r)\right| \\
&= \left|g(\tilde{u}) - g(u^-)\right| + \left|g(u^-) - g(u_l)\right| \\
&= |g(\tilde{u}) - g(u_l)| \\
&= \text{Tot.Var.}\ [f(u(\bar{t}-,\cdot)) + g(u(\bar{t}-,\cdot))]\,,
\end{aligned}$$

where we used (H1) and (H4).

Suppose now that a wave $(u_r, \tilde{u})$ with negative speed interacts with J from the outgoing edge I_2. We denote with (u^-, u^+) the solution to the Riemann problem at J with the initial datum $(u_l, \tilde{u})$. We have

$$\begin{aligned}
\text{Tot.Var.}\ [g(u(\bar{t}+,\cdot)) + f(u(\bar{t}+,\cdot))] &= \left|g(u_l) - g(u^-)\right| + \left|f(u^+) - f(\tilde{u})\right| \\
&= \left|f(u_r) - f(u^+)\right| + \left|f(u^-) - f(\tilde{u})\right| \\
&= |f(\tilde{u}) - f(u_r)| \\
&= \text{Tot.Var.}\ [f(u(\bar{t}-,\cdot)) + g(u(\bar{t}-,\cdot))]\,,
\end{aligned}$$

where we used (H1) and (H5).

This completes the proof. □

Theorem 4.4.15. *Consider the network composed by the incoming road I_1, the outgoing one I_2 and by the junction J. Fix an admissible Riemann solver RS at J. For every $t \geqslant 0$, it holds*

$$\textit{Tot.Var.}(f(u(t,\cdot))) \leqslant \textit{Tot.Var.}(f(u(0+,\cdot))). \tag{4.4.13}$$

Proof. By Lemma 4.4.14, we know that the total variation of the flux does not change when a wave approaches the vertex J. If, instead, two waves interact in a edge, then the total variation of the flux either remains constant or strictly decreases. This completes the proof. □

4.5 Exercises

Exercise 4.5.1. Consider a scalar conservation law with flux function given by:

$$f(\rho) = \begin{cases} \rho(1-\rho), & \text{if } 0 \leqslant \rho \leqslant \frac{1}{2}, \\ \frac{1}{4}, & \text{if } \frac{1}{2} \leqslant \rho \leqslant 1, \\ -\rho^2 + 2\rho - \frac{3}{4}, & \text{if } 1 \leqslant \rho \leqslant \frac{3}{2}. \end{cases}$$

Fix a junction and an initial datum $\rho_0 \in [0, \frac{3}{2}]$ on an incoming road I_i. Find all possible values taken by a Riemann solver, respecting **(Cons.1)** and **(Cons.2)** of Section 4.2, on I_i.

Exercise 4.5.2. Consider a scalar conservation law satisfying (H) of Section 4.3.1, a road $I = [a, b]$ on a network and assume that the initial datum on I is given by

$$\rho_0(x) = \begin{cases} 0, & \text{if } x \leqslant c, \\ \rho_{max}, & \text{if } x > c, \end{cases}$$

where $c \in]a, b[$. Determine how many big shocks may be present on I for any positive time.

Exercise 4.5.3. Consider a 1-1 vertex as in Section 4.4.
Does it exist a Riemann solver satisfying (H1) and (H2) of Section 4.4.1 such that the map $(u_l, u_r) \mapsto RS(u_l, u_r)$ is continuous?
Does it exist a Riemann solver satisfying (H1) and (H2) of Section 4.4.1 and providing uniqueness of weak solutions such that the map $(u_l, u_r) \mapsto RS(u_l, u_r)$ is continuous?

Exercise 4.5.4. Consider a 1-1 vertex as in Section 4.4, but now a rich system of conservation laws on each road. Find bounds on the number of waves produced along a wave-front tracking approximate solution.

Exercise 4.5.5. Consider a 1-1 vertex as in Section 4.4, with the same assumptions of Section 4.4.1. Assume that the set X, defined in (4.4.11), is a singleton $\bar{\gamma}$. Prove that if $\bar{\gamma} < \min\{g(\sigma_g), f(\sigma_f)\}$, where σ_g and σ_f are defined at the beginning of Section 4.4, then the total variation of the density remains bounded along wave-front tracking approximate solutions (for initial data with bounded variation).

4.6 Open Problems

Problem 4.6.1. Prove existence of solutions on a network to a hyperbolic system of conservation laws, which is not rich. More precisely, assume that it exists a region D invariant for solutions to Riemann problems both on edges and on vertices. Under what hypotheses is it possible to construct a sequence of wave-front tracking approximate solutions converging to a weak entropic solution?

Problem 4.6.2. Construct all possible Riemann solvers, analogous to those of Section 4.4.1, in the general case of a vertex with n incoming and m outgoing edges.

5

Lighthill-Whitham-Richards Model on Networks

This chapter deals with a road network, which is a network where each edge and each vertex represents respectively an unidirectional road and a junction. On each road we consider the Lighthill-Whitham-Richards model for traffic, while at junctions we consider a Riemann solver satisfying the conservation of cars (recall conditions **Cons.** of Section 4.2) and the following rules:

(A) there are some prescribed preferences of drivers, that is the traffic from incoming roads is distributed on outgoing roads according to fixed coefficients;
(B) respecting (A), drivers choose so as to maximize fluxes.

In the case of junctions with 2 incoming roads and 1 outgoing one (or in general with more incoming than outgoing roads), it is necessary to introduce right of way parameters, which determine the priority among incoming roads.

5.1 Basic Definitions and Assumptions

Definition 5.1.1. *A road network is a network, as in Definition 4.1.1, in which the edges and the vertices represent respectively unidirectional roads and junctions.*

Fix a road network $(\mathcal{I}, \mathcal{J})$. On each road consider the equation

$$\rho_t + f(\rho)_x = 0, \tag{5.1.1}$$

where $\rho = \rho(t, x) \in [0, \rho_{max}]$, $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$, is the *density* of cars, v is the *average speed* and $f(\rho) = v\,\rho$ is the *flux*. We assume the following:

(A1) $\rho_{max} = 1$;
(A2) the speed v depends only on the density ρ;
(A3) the flux f is a strictly concave C^2 function;
(A4) $f(0) = f(1) = 0$.

Notice that the previous assumptions (A3) and (A4) imply that f has a unique point of maximum $\sigma \in]0,1[$.

Remark 5.1.2. The regularity of the flux function can be relaxed. In fact, the following analysis works for Lipschitz continuous functions.

Remark 5.1.3. The case of not strictly concave fluxes can be treated under suitable assumptions. More precisely, the analysis can be carried out in the same way if the analysis of Section 4.3.1, and in particular Proposition 4.3.3, is still valid.
This happens, for instance, if assumptions (NSC1)–(NSC3) of Section 3.1.4 hold true.

At each junction J, it is given a traffic-distribution matrix, i.e a matrix describing the distribution of the traffic among outgoing roads.

Definition 5.1.4. *Fix a vertex J with n incoming edges, say $I_1, \ldots, I_n$, and m outgoing edges, say $I_{n+1}, \ldots, I_{n+m}$. A traffic distribution matrix A is given by*

$$A = \begin{pmatrix} \alpha_{n+1,1} & \cdots & \alpha_{n+1,n} \\ \vdots & \vdots & \vdots \\ \alpha_{n+m,1} & \cdots & \alpha_{n+m,n} \end{pmatrix}, \tag{5.1.2}$$

where $0 \leqslant \alpha_{j,i} \leqslant 1$ for every $i \in \{1, \ldots, n\}$ and for every $j \in \{n+1, \ldots, n+m\}$ and

$$\sum_{j=n+1}^{n+m} \alpha_{j,i} = 1 \tag{5.1.3}$$

for every $i \in \{1, \ldots, n\}$.

Given a junction J and an incoming road I_i, the i-th column of A describes how the traffic from I_i distributes in percentages to the outgoing roads. This means that if C is the quantity of traffic coming from road I_i then $\alpha_{j,i}C$ traffic moves towards roads I_j.

Remark 5.1.5. One may also assume that the matrix A is time dependent. For example in case of car traffic on an urban network, the preferences of drivers may change depending on the period of the day.

We introduce a technical condition on matrix A. We say that the matrix A satisfies hypothesis (C) if the following holds.

(C) Let $\{e_1, \ldots, e_n\}$ be the canonical basis of $\mathbb{R}^n$ and for every subset $V \subset \mathbb{R}^n$ indicate by $V^\perp$ its orthogonal. Define for every $i = 1, \ldots, n$, $H_i = \{e_i\}^\perp$, i.e. the coordinate hyperplane orthogonal to e_i and, for every $j = n+1, \ldots, n+m$ let, $\alpha_j = (\alpha_{j1}, \ldots, \alpha_{jn}) \in \mathbb{R}^n$ and define $H_j = \{\alpha_j\}^\perp$. Let $\mathcal{K}$ be the set of indices $k = (k_1, ..., k_\ell)$, $1 \leq \ell \leq n-1$, such that $0 \leq k_1 <$

$k_2 < \cdots < k_\ell \leq n+m$ and for every $k \in \mathcal{K}$ set $H_k = \bigcap_{h=1}^{\ell} H_{k_h}$. Letting $\mathbf{1} = (1, \ldots, 1) \in \mathbb{R}^n$, then for every $k \in \mathcal{K}$,

$$\mathbf{1} \notin H_k^{\perp}. \tag{5.1.4}$$

Remark 5.1.6. Condition (C) is a technical condition, which is important to isolate a unique solution to Riemann problems at junctions. From (C) we immediately derive $m \geqslant n$. Otherwise, since by definition $\mathbf{1} = \sum_{j=n+1}^{n+m} \alpha_j$, we get $\mathbf{1} \in H_k^{\perp}$, where

$$H_k = \cap_{j=n+1}^{n+m} H_j.$$

In case $m = n$ it is easy to check that condition (C) is generic in the space of $n \times n$ matrices, which means that the set of matrices satisfying (C) is open and dense.

Moreover if $n \geqslant 2$, then (C) implies that, for every $j \in \{n+1, \ldots, n+m\}$ and for every distinct elements $i, i' \in \{1, \ldots, n\}$, it holds $\alpha_{j,i} \neq \alpha_{j,i'}$. Otherwise, without loss of generalities, we may suppose that $\alpha_{n+1,1} = \alpha_{n+1,2}$. If we consider

$$H = (\cap_{2<j\leqslant n} H_j) \cap H_{n+1},$$

then, by (C), there exists an element $(x_1, x_2, 0, \ldots, 0) \in H$ such that $x_1 + x_2 \neq 0$ and $\alpha_{n+1,1}(x_1 + x_2) = 0$.

In the case of a simple vertex J with 2 incoming edges and 2 outgoing ones, the condition (C) is completely equivalent to the fact that, for every $j \in \{3, 4\}$, $\alpha_{j,1} \neq \alpha_{j,2}$.

Remark 5.1.7. Notice that a distribution matrix A satisfying condition (C) could have identical lines. For example

$$A = \begin{pmatrix} \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{3} & \frac{1}{2} & \frac{3}{5} \end{pmatrix}$$

satisfies the condition (C).

Assume that each traffic-distribution matrix satisfies hypothesis (C).

Denote with $\rho_i : [0, +\infty[\times I_i \to [0, 1]$ the density of cars in the road I_i of the network. We want ρ_i to be a weak entropic solution on I_i, i.e. for every function $\varphi : [0, +\infty[\times I_i \to \mathbb{R}$ smooth with compact support on $]0, +\infty[\times]a_i,$

$$\int_0^{+\infty} \int_{a_i}^{b_i} \left(\rho_i \frac{\partial \varphi}{\partial t} + f(\rho_i) \frac{\partial \varphi}{\partial x} \right) dx dt = 0, \tag{5.1.5}$$

and for every $k \in \mathbb{R}$ and every $\tilde{\varphi} : [0, +\infty[\times I_i \to \mathbb{R}$ smooth, positive with compact support on $]0, +\infty[\times]a_i, b_i[$

$$\int_0^{+\infty}\int_{a_i}^{b_i}\Big(|\rho_i-k|\frac{\partial\tilde{\varphi}}{\partial t}+\text{sgn }(\rho_i-k)(f(\rho_i)-f(k))\frac{\partial\tilde{\varphi}}{\partial x}\Big)dxdt\geq 0; \quad (5.1.6)$$

see Chapter 2.

In the sequel, we are now ready to give the definitions of solution at junctions and on the whole network.

Definition 5.1.8. *Let J be a junction with incoming roads, say $I_1,\ldots,I_n$, and outgoing roads, say $I_{n+1},\ldots,I_{n+m}$. A weak solution at J is a collection of functions $\rho_l : [0,+\infty[\times I_l \to \mathbb{R}$, $l = 1,\ldots,n+m$, such that*

$$\sum_{l=1}^{n+m}\Big(\int_0^{+\infty}\int_{a_l}^{b_l}\Big(\rho_l\frac{\partial\varphi_l}{\partial t}+f(\rho_l)\frac{\partial\varphi_l}{\partial x}\Big)dxdt\Big)=0, \quad (5.1.7)$$

for every φ_l, $l = 1,\ldots,n+m$ smooth having compact support in the set $]0,+\infty[\times]a_l,b_l]$ for $l = 1,\ldots,n$ (incoming roads) and in $]0,+\infty[\times[a_l,b_l[$ for $l = n+1,\ldots,n+m$ (outgoing roads), that are also smooth across the junction, *i.e.*

$$\varphi_i(\cdot,b_i)=\varphi_j(\cdot,a_j), \quad \frac{\partial\varphi_i}{\partial x}(\cdot,b_i)=\frac{\partial\varphi_j}{\partial x}(\cdot,a_j),$$

where $i \in \{1,...,n\}$ and $j \in \{n+1,...,n+m\}$.

Lemma 5.1.9. *Let $\rho = (\rho_1,\ldots,\rho_{n+m})$ be a weak solution at the junction such that each $x \to \rho_i(t,x)$ has bounded variation. Then ρ satisfies the Rankine-Hugoniot Condition at the junction J, namely*

$$\sum_{i=1}^{n} f(\rho_i(t,b_i-))=\sum_{j=n+1}^{n+m} f(\rho_j(t,a_j+)), \quad (5.1.8)$$

for almost every $t > 0$.

Proof. The proof is similar to the proof of Theorem 2.2.5 Suppose for simplicity that, for every $l \in \{1,\ldots,n+m\}$, ρ_l is constant on I_l. Then (5.1.7) implies that

$$\sum_{l=1}^{n+m}\int_0^{+\infty}\int_{a_l}^{b_l}\text{div }(\rho_l\varphi_l,f(\rho_l)\varphi_l)\,dxdt=0.$$

By applying the divergence theorem to the last expression and by using the hypotheses on the functions φ_l we get

$$\int_0^{+\infty}\left(\sum_{l=1}^{n}f(\rho_l(t,b_l))-\sum_{l=n+1}^{n+m}f(\rho_l(t,a_l))\right)\varphi_1(t,b_l)dt=0$$

and so

$$\sum_{l=1}^{n}f(\rho_l(t,b_l))=\sum_{l=n+1}^{n+m}f(\rho_l(t,a_l))$$

by the arbitrariness of the function φ_1. □

Definition 5.1.10. *Let $\rho = (\rho_1, \ldots, \rho_{n+m})$ be such that $\rho_i(t, \cdot)$ is of bounded variation for every $t \geqslant 0$. Then ρ is an admissible weak solution of (5.1.1) related to the matrix A at the junction J if and only if the following properties hold:*

(i) ρ is a weak solution at the junction J;

(ii) $f(\rho_j(\cdot, a_j+)) = \sum_{i=1}^{n} \alpha_{j,i} f(\rho_i(\cdot, b_i-))$, for each $j = n+1, ..., n+m$;

(iii) $\sum_{i=1}^{n} f(\rho_i(\cdot, b_i-))$ is maximum subject to (i) and (ii).

Remark 5.1.11. The assumption (i) of the previous Definition is essentially the conservation of car at junctions, as seen in Lemma 5.1.9 Assumptions (ii) and (iii), instead, describe the rules (A) and (B), i.e. the preferences of drivers and the maximization procedure.

Definition 5.1.12. *Given $\bar{\rho}_i : I_i \to \mathbb{R}$, $i = 1, \ldots, N$, L^∞ functions, a collection of functions $\rho = (\rho_1, \ldots, \rho_N)$, with $\rho_i : [0, +\infty[\times I_i \to \mathbb{R}$ continuous as functions from $[0, +\infty[$ into L^1_{loc}, is an admissible solution if ρ_i is a weak entropic solution to (5.1.1) on I_i, $\rho_i(0, x) = \bar{\rho}_i(x)$ a.e., at each junction ρ is a weak solution and is an admissible weak solution in case of bounded variation.*

Remark 5.1.13. According to Definition 4.1.1, for every road $I_i = [a_i, b_i]$ of the network, we suppose that, if $a_i > -\infty$, then it is an incoming road for a junction, while if $b_i < +\infty$, then it is an outgoing road for a junction. Then a solution for every time is determined just by initial data on the network.

In the general case, i.e. when there are roads with a_i finite but not outgoing for any junction (or b_i finite but not incoming for any junction), we have to assign also boundary data, in the sense of [2, 5, 15], and all next results hold with obvious modifications.

5.2 The Riemann Problem at Junctions

In this section we construct step by step a particular Riemann solver at junctions satisfying rules (A) and (B). We also construct a Riemann solver satisfying a precedence rule in the case of junctions with the number of outgoing roads bigger than the number of incoming ones. We treat with special attention the case of junctions with one incoming road and two outgoing roads.

Consider a junction J with n incoming roads and m outgoing roads (see Figure 5.1) and a distribution matrix A. For simplicity we indicate by

$$(t, x) \in \mathbb{R}_+ \times I_i \mapsto \rho_i(t, x) \in [0, 1], \quad i = 1, ..., n, \tag{5.2.9}$$

the densities of the cars on the roads with incoming traffic and

$$(t, x) \in \mathbb{R}_+ \times I_j \mapsto \rho_j(t, x) \in [0, 1], \quad j = n+1, ..., n+m \tag{5.2.10}$$

those on the roads with outgoing traffic. Suppose that $(\rho_{1,0}, \ldots, \rho_{n+m,0})$ are the initial densities in each road of the junction J. In this section, we consider the function τ of Definition 4.3.1

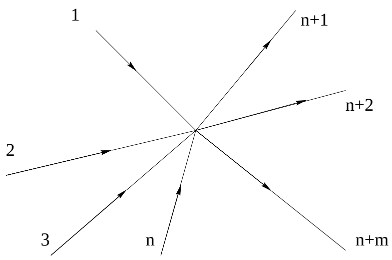


Fig. 5.1. a junction with n incoming roads and m outgoing roads.

5.2.1 The Case $n \leqslant m$

Assume that $n \leqslant m$, i.e. the number of outgoing roads is greater than or equal to the number of incoming roads. The next theorem gives the unique Riemann solver RS producing admissible weak solution to a Riemann problem at the junction J satisfying rules (A) and (B).

Theorem 5.2.1. *Consider a junction J, assume (A1)-(A4) and that the matrix A satisfies condition (C). For every $\rho_{1,0}, \ldots, \rho_{n+m,0} \in [0,1]$, there exists a unique admissible centered weak solution $\rho = (\rho_1, \ldots, \rho_{n+m})$ to (5.1.1) at the junction J, in the sense of Definition 5.1.10, such that*

$$\rho_1(0,\cdot) \equiv \rho_{1,0}, \ldots, \rho_{n+m}(0,\cdot) \equiv \rho_{n+m,0}.$$

Moreover, there exists a unique $(n+m)-$tuple $(\hat{\rho}_1, ..., \hat{\rho}_{n+m}) \in [0,1]^{n+m}$ such that

$$\hat{\rho}_i \in \begin{cases} \{\rho_{i,0}\} \cup]\tau(\rho_{i,0}), 1], & \text{if } 0 \leqslant \rho_{i,0} \leqslant \sigma, \\ [\sigma, 1], & \text{if } \sigma \leqslant \rho_{i,0} \leqslant 1, \end{cases} \qquad i = 1, ..., n, \tag{5.2.11}$$

and

$$\hat{\rho}_j \in \begin{cases} [0, \sigma], & \text{if } 0 \leqslant \rho_{j,0} \leqslant \sigma, \\ \{\rho_{j,0}\} \cup [0, \tau(\rho_{j,0})[, & \text{if } \sigma \leqslant \rho_{j,0} \leqslant 1, \end{cases} \qquad j = n+1, ..., n+m, \tag{5.2.12}$$

and for $i \in \{1, \ldots, n\}$ the solution is given by the wave $(\rho_{i,0}, \hat{\rho}_i)$, while for $j \in \{n+1, \ldots, n+m\}$ the solution is given by the wave $(\hat{\rho}_j, \rho_{j,0})$.

The next corollary is an easy consequence of the previous theorem.

Corollary 5.2.2. *Consider a junction J, assume (A1)-(A4) and that the matrix A satisfies condition (C). There exists a unique Riemann solver RS compatible with Definition 5.1.10. Moreover, for every $\rho_{1,0}, \ldots, \rho_{n+m,0} \in [0,1]$, the $n+m$-tuple $(\hat{\rho}_1, ..., \hat{\rho}_{n+m}) = RS(\rho_{1,0}, \ldots, \rho_{n+m,0})$ satisfies (5.2.11) and (5.2.12).*

Proof of Theorem 5.2.1 Define the map

$$E : (\gamma_1, ..., \gamma_n) \in \mathbb{R}^n \longmapsto \sum_{i=1}^{n} \gamma_i \tag{5.2.13}$$

and the sets

$$\begin{aligned} \Omega_i &:= [0, \gamma_i^{max}(\rho_{i,0})], \quad i = 1, ..., n, \\ \Omega_j &:= \left[0, \gamma_j^{max}(\rho_{j,0})\right], \quad j = n+1, ..., n+m, \\ \Omega &:= \left\{(\gamma_1, \ldots, \gamma_n) \in \Omega_1 \times \cdots \times \Omega_n | A \cdot (\gamma_1, ..., \gamma_n)^T \in \Omega_{n+1} \times \cdots \times \Omega_{n+m}\right\}, \end{aligned} \tag{5.2.14}$$

where the functions γ_i^{max} and γ_j^{max} are respectively defined in (4.3.3) and in (4.3.4).

By Proposition 4.3.3, the sets Ω_i, Ω_j contain all the possible fluxes for the solution to the Riemann problem at J. The set Ω is closed, convex and not empty; see Figure 5.2.

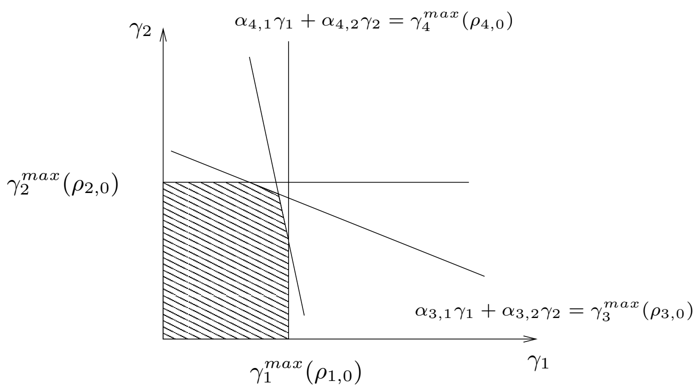


Fig. 5.2. the set Ω for a simple junction J with 2 incoming and 2 outgoing roads.

Moreover, by (C), $\nabla E = \mathbf{1}$ is not orthogonal to any nontrivial subspace contained in a supporting hyperplane of Ω, hence there exists a unique vector $(\hat{\gamma}_1, \ldots, \hat{\gamma}_n) \in \Omega$ such that

$$E(\hat{\gamma}_1, \ldots, \hat{\gamma}_n) = \max_{(\gamma_1, \ldots, \gamma_n) \in \Omega} E(\gamma_1, \ldots, \gamma_n).$$

For every $i \in \{1, \ldots, n\}$, we choose $\hat{\rho}_i \in [0,1]$ such that

$$f(\hat{\rho}_i) = \hat{\gamma}_i, \quad \hat{\rho}_i \in \begin{cases} \{\rho_{i,0}\} \cup]\tau(\rho_{i,0}), 1], & \text{if } 0 \leqslant \rho_{i,0} \leqslant \sigma, \\ [\sigma, 1], & \text{if } \sigma \leqslant \rho_{i,0} \leqslant 1. \end{cases}$$

By (A3) and (A4), $\hat{\rho}_i$ exists and is unique. Let

$$\hat{\gamma}_j \doteq \sum_{i=1}^{n} \alpha_{ji} \hat{\gamma}_i, \qquad j = n+1, \dots, n+m,$$

and $\hat{\rho}_j \in [0, 1]$ be such that

$$f(\hat{\rho}_j) = \hat{\gamma}_j, \quad \hat{\rho}_j \in \begin{cases} [0, \sigma], & \text{if } 0 \leqslant \rho_{j,0} \leqslant \sigma, \\ \{\rho_{j,0}\} \cup [0, \tau(\rho_{j,0})[, & \text{if } \sigma \leqslant \rho_{j,0} \leqslant 1. \end{cases}$$

Since $(\hat{\gamma}_1, \dots, \hat{\gamma}_n) \in \Omega$, $\hat{\rho}_j$ exists and is unique for every $j \in \{n+1, \dots, n+m\}$. Solving the Riemann Problem (see Chapter 2) on each road, the claim is proved. □

Example 5.2.3. Fix a junction J with 2 incoming roads I_1 and I_2 and with 2 outgoing roads I_3 and I_4. Assume that the distribution matrix A for J takes the form

$$A = \begin{pmatrix} \frac{1}{3} & \frac{1}{4} \\ \frac{2}{3} & \frac{3}{4} \end{pmatrix}.$$

Consider the initial data $(\rho_{1,0}, \rho_{2,0}, \rho_{3,0}, \rho_{4,0})$ such that

$$0 < \rho_{1,0} < \sigma, \quad \sigma < \rho_{2,0} < 1, \quad 0 < \rho_{3,0} < \sigma, \quad \sigma < \rho_{4,0} < 1,$$
$$f(\rho_{1,0}) = f(\rho_{4,0}) = \frac{1}{2},$$

and suppose that $f(\sigma) = 1$. Using the same notation of the proof of Theorem 5.2.1 we get

$$\Omega_1 = \Omega_4 = \left[0, \frac{1}{2}\right], \quad \Omega_2 = \Omega_3 = [0, 1],$$

where the sets Ω_i describes the possible values for the fluxes in each road of the junction. The set

$$\Omega = \left\{(\gamma_1, \gamma_2) \in \Omega_1 \times \Omega_2 | A \cdot (\gamma_1, \gamma_2)^T \in \Omega_3 \times \Omega_4\right\}$$

is described in Figure 5.3.

The solution for the fluxes is given by

$$\left(\frac{1}{2}, \frac{2}{9}, \frac{2}{9}, \frac{1}{2}\right),$$

and so

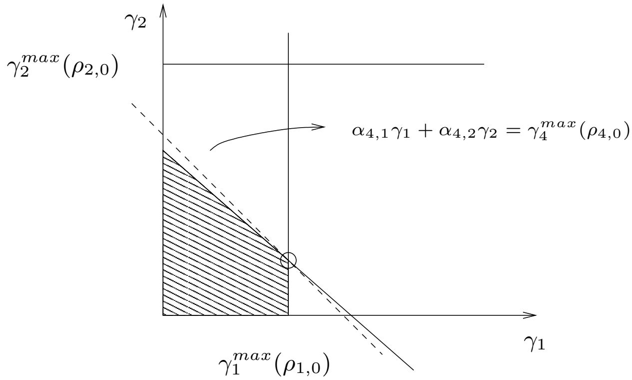


Fig. 5.3. the set Ω of Example 5.2.3

$$\hat{\rho}_1 = \rho_{1,0}, \quad \hat{\rho}_2 > \sigma, \quad f(\hat{\rho}_2) = \frac{2}{9},$$
$$\hat{\rho}_3 < \sigma, \quad f(\hat{\rho}_3) = \frac{2}{9}, \quad \hat{\rho}_4 = \rho_{4,0}.$$

The solution to the Riemann problem at J is completely determined.

Remark 5.2.4. Notice that the case of junction with two incoming roads and one outgoing road or, more generally, the case on n incoming roads and one outgoing road is not covered by the previous theorem.

Remark 5.2.5. Notice that in the proof of Theorem 5.2.1, a solution to the Riemann problem is obtained solving a Linear Programming (briefly LP) problem on the incoming fluxes. More precisely, we find $\hat{\gamma} = (\hat{\gamma}_1, \ldots, \hat{\gamma}_n)$ as solution to:

$$\begin{aligned} \max_{\hat{\gamma}} \quad & E(\hat{\gamma}) \\ \hat{\gamma} &\in \Omega_1 \times \cdots \times \Omega_n \\ A \cdot \hat{\gamma}^T &\in \Omega_{n+1} \times \cdots \times \Omega_{n+m} \end{aligned} \tag{5.2.15}$$

where E is defined in (5.2.13) and the sets Ω_i and Ω_j in (5.2.14). Condition (C) is precisely the necessary condition to have uniqueness of a solution to (5.2.15).

5.2.2 The Case of $n \geqslant 2$ Incoming Roads and $m = 1$ Outgoing Road

Condition (C) on A can not hold for crossings with n incoming roads and one outgoing road. We thus introduce some further parameters whose meaning is the following. When not all cars can go through the junction, there is a yielding rule that describes the percentage of cars crossing the junction, which comes from a particular incoming road.

Consider first the case $n = 2$. To treat this case we fix a *right of way* parameter $q \in]0, 1[$ and assign the rule:

(P) Assume that not all cars can enter the outgoing road and let C be the amount that can do it. Then qC cars come from first incoming road and $(1 - q)C$ cars from the second.

Let us fix a crossing with two incoming roads $[a_i, b_i]$, $i = 1, 2$, and one outgoing road $[a_3, b_3]$ and assume that a right of way parameter $q \in]0, 1[$ is given. Then the solution to the Riemann problem $(\rho_{1,0}, \rho_{2,0}, \rho_{3,0})$ is formed by a single wave on each road connecting the initial states to $(\hat\rho_1, \hat\rho_2, \hat\rho_3)$ determined in the following way.

Since we want to maximize going through traffic we set:

$$\hat\gamma_3 = \min\{\gamma_1^{max}(\rho_{1,0}) + \gamma_2^{max}(\rho_{2,0}), \gamma_3^{max}(\rho_{3,0})\}, \tag{5.2.16}$$

where the functions γ_i^{max} are defined in (4.3.3) and in (4.3.4). Consider the space (γ_1, γ_2) and the line:

$$\gamma_2 = \frac{1 - q}{q}\gamma_1. \tag{5.2.17}$$

Notice that the line is exactly the locus of points satisfying exactly rule (P). Define P to be the point of intersection of the line (5.2.17) with the line $\gamma_1 + \gamma_2 = \hat\gamma_3$. Recall that the final fluxes should belong to the region

$$\Omega = \{(\gamma_1, \gamma_2) : 0 \leqslant \gamma_i \leqslant \gamma_i^{max}(\rho_{i,0}), 0 \leqslant \gamma_1 + \gamma_2 \leqslant \hat\gamma_3\}.$$

We distinguish two cases:

a) P belongs to Ω,
b) P is outside Ω.

In the first case we set $(\hat\gamma_1, \hat\gamma_2) = P$, while in the second we set $(\hat\gamma_1, \hat\gamma_2) = Q$, where Q is the point of the segment $\Omega \cap \{(\gamma_1, \gamma_2) : \gamma_1 + \gamma_2 = \hat\gamma_3\}$ closest to the line (5.2.17). We show in Figure 5.4 the two cases.

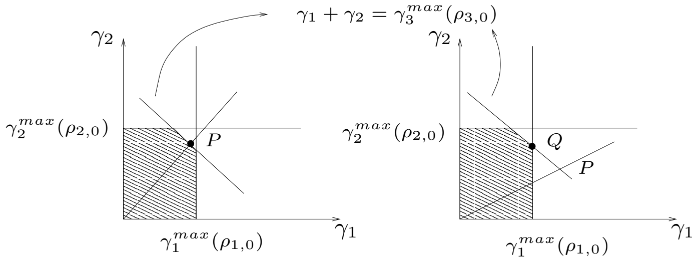


Fig. 5.4. The cases a) and b).

Remark 5.2.6. Note that in the second case b) it is not possible to respect in an exact way the rule (P) if we want also to maximize the flux. So the point Q is the point that approximates better the rule (P) in the set of points that maximize the sum of the fluxes.

Once we have determined $\hat{\gamma}_1$ and $\hat{\gamma}_2$ (hence also $\hat{\gamma}_3$) we can determine in a unique way $\hat{\rho}_i$ $(i \in \{1,2,3\})$. Therefore the following theorem holds.

Theorem 5.2.7. *Consider a junction J with $n = 2$ incoming roads and $m = 1$ outgoing road, assume (A1)-(A4) and fix the right of way parameter $q \in]0,1[$. For every $\rho_{1,0}, \rho_{2,0}, \rho_{3,0} \in [0,1]$, there exists a unique admissible centered weak solution $\rho = (\rho_1, \rho_2, \rho_3)$ to (5.1.1) at the junction J, in the sense of Definition 5.1.10, satisfying rule (P) (possibly in a approximate way) such that*

$$\rho_1(0,\cdot) \equiv \rho_{1,0},\ \rho_2(0,\cdot) \equiv \rho_{2,0},\ \rho_3(0,\cdot) \equiv \rho_{3,0}.$$

Moreover, there exists a unique $3-$*tuple* $(\hat{\rho}_1, \hat{\rho}_2, \hat{\rho}_3) \in [0,1]^3$ *such that*

$$\hat{\rho}_i \in \begin{cases} \{\rho_{i,0}\} \cup]\tau(\rho_{i,0}), 1], & \text{if } 0 \leqslant \rho_{i,0} \leqslant \sigma, \\ [\sigma, 1], & \text{if } \sigma \leqslant \rho_{i,0} \leqslant 1, \end{cases} \qquad i = 1,2, \tag{5.2.18}$$

and

$$\hat{\rho}_3 \in \begin{cases} [0, \sigma], & \text{if } 0 \leqslant \rho_{3,0} \leqslant \sigma, \\ \{\rho_{3,0}\} \cup [0, \tau(\rho_{3,0})[, & \text{if } \sigma \leqslant \rho_{3,0} \leqslant 1, \end{cases} \tag{5.2.19}$$

and for $i \in \{1,2\}$ the solution is given by the wave $(\rho_{i,0}, \hat{\rho}_i)$, while for the outgoing road the solution is given by the wave $(\hat{\rho}_3, \rho_{3,0})$.

Corollary 5.2.8. *Consider a junction J with $n = 2$ incoming roads and $m = 1$ outgoing road, assume (A1)-(A4) and fix the right of way parameter $q \in]0,1[$. Then there exists a unique Riemann solver RS, compatible with Definition 5.1.10 and rule (P). Moreover, for every $\rho_{1,0}, \rho_{2,0}, \rho_{3,0} \in [0,1]$, the 3-tuple $(\hat{\rho}_1, \hat{\rho}_1, \hat{\rho}_1) = RS(\rho_{1,0}, \rho_{2,0}, \rho_{3,0})$ satisfies (5.2.18) and (5.2.19).*

We briefly describe now the case of a junction J with $n > 2$ incoming roads and $m = 1$ outgoing road. Fix $n-1$ positive parameters $q_1, \ldots, q_{n-1}$ and consider the line r in $\mathbb{R}^n$, given by

$$\begin{cases} \gamma_n = q_1 \gamma_1, \\ \qquad \vdots \\ \gamma_n = q_{n-1} \gamma_{n-1}. \end{cases} \tag{5.2.20}$$

Then the solution to the Riemann problem with initial conditions given by $(\rho_{1,0}, \ldots, \rho_{n,0}, \rho_{n+1,0})$ is formed by waves connecting the initial states to $(\hat{\rho}_1, \ldots, \hat{\rho}_n, \hat{\rho}_{n+1})$, determined in the following way. As in the previous case, define

$$\hat{\gamma}_{n+1} = \min\left\{\gamma_1^{max}(\rho_{1,0}) + \cdots + \gamma_n^{max}(\rho_{n,0}), \gamma_{n+1}^{max}(\rho_{n+1,0})\right\} \tag{5.2.21}$$

where the functions γ_i^{max} are defined in (4.3.3) and in (4.3.4). Define the closed and convex set K in $\mathbb{R}^n$

$$\{(\gamma_1, \ldots, \gamma_n) : \gamma_1 + \cdots + \gamma_n = \hat{\gamma}_{n+1},\, 0 \leqslant \gamma_i \leqslant \gamma_i^{max}(\rho_{i,0}),\, i = 1, \ldots, n\} \tag{5.2.22}$$

Consider the unique point $(\hat{\gamma}_1, \ldots, \hat{\gamma}_n) \in K$ which minimizes the distance from the point $P \in r$, where P is the intersection between the line r and the hyperplane

$$\gamma_1 + \cdots + \gamma_n = \hat{\gamma}_{n+1}.$$

Finally imposing $f(\hat{\rho}_l) = \hat{\gamma}_l$ $(l = 1, \ldots, n, n+1)$, we obtain the trace of the solution to the Riemann problem at the junction.

5.2.3 Demand-Supply of Lebacque

An interesting point of view is that of Demand-Supply introduced by J.P. Lebacque.

We can define the demand function as

$$D(\rho) = \begin{cases} f(\rho), & \text{if } \rho \in [0, \sigma], \\ f(\sigma), & \text{if } \rho \in]\sigma, 1], \end{cases}$$

and the supply function as

$$S(\rho) = \begin{cases} f(\sigma), & \text{if } \rho \in [0, \sigma], \\ f(\rho), & \text{if } \rho \in]\sigma, 1]. \end{cases}$$

The demand function measures the maximum flux that an incoming road may demand to send, while the supply function measures the maximum flux that an outgoing road may supply space for. Reasoning on demand-supply functions, it is possible to define various Riemann solvers, for example that respecting rules (A) and (B). In this case one would choose the maximum demands respecting rule (A).

The connection between the two theory is clarified by the following formulas. Fix a junction J, with n incoming and m outgoing road, and a Riemann data $(\rho_{1,0}, \ldots, \rho_{n+m,0})$. Then, for every incoming road I_i, $i = 1, \ldots, n$, one has:

$$D(\rho_{i,0}) = \gamma_i^{max}$$

and for every outgoing road I_j, $j = n+1, \ldots, n+m$:

$$S(\rho_{j,0}) = \gamma_j^{max}.$$

5.3 Estimates on Flux Variation and Existence of Solutions

This section is devoted to construct solutions to the Cauchy problem. This is achieved by estimating the number of waves and interactions, and the total variation of the flux along an approximate wave front tracking solution. From now on, we assume that

(H1) every junction has exactly two incoming roads and two outgoing ones.

This hypothesis is crucial, because, as shown in Section 5.6, the presence of more complicate junctions provokes additional increases of the total variation of the flux. So, for each junction J, the distributional matrix A takes the form

$$A = \begin{pmatrix} \alpha & \beta \\ 1-\alpha & 1-\beta \end{pmatrix}, \tag{5.3.23}$$

where $\alpha, \beta \in]0,1[$ and $\alpha \neq \beta$, so that (C) is satisfied.

Remark 5.3.1. We prove all the next estimates in the case of a junction with 2 incoming roads and 2 outgoing ones. However it is possible to extend the results to the following cases.

1. Junctions with 2 incoming roads and 1 outgoing one.
2. Junctions with 1 incoming road and 2 outgoing ones.
3. Junctions with 1 incoming road and 1 outgoing one.

5.3.1 Estimates on the Number of Waves and Interactions

We now prove that assumption (H*) of Section 4.3.1 holds for junctions with two incoming and two outgoing roads. Recall the definitions and results of Section 4.3.1.

From now on we fix an approximate wave front tracking solution ρ, defined on the road network. We fist need some notation.

Definition 5.3.2. *For every road I_i, $i = 1, \ldots, N$, we indicate by*

$$(\rho_-^\theta, \rho_+^\theta), \quad \theta \in \Theta = \Theta(\rho, t, i), \quad \Theta \text{ finite set,}$$

the discontinuities on road I_i at time t, and by $x^\theta(t)$, $\lambda^\theta(t)$, $\theta \in \Theta$, respectively their positions and velocities at time t. We also refer to the wave θ to indicate the discontinuity $(\rho_-^\theta, \rho_+^\theta)$.

For each discontinuity $(\rho_-^\theta, \rho_+^\theta)$ at time $\bar{t}$ on road I_i, we call $y^\theta(t)$, $t \in [\bar{t}, t_\theta]$, the trace of the wave so defined. We start with $y^\theta(\bar{t}) = x^\theta(\bar{t})$ and we continue up to the first interaction with another wave or a junction. If at time $\tilde{t}$ an interaction with a wave or a junction occurs, then either a single new wave $(\rho_-^{\tilde{\theta}}, \rho_+^{\tilde{\theta}})$ on road I_i is produced or no wave is produced. In the latter case we set $t_\theta = \tilde{t}$, otherwise we set $y^\theta(\tilde{t}) = x^{\tilde{\theta}}(\tilde{t})$ and follow $x^{\tilde{\theta}}(t)$ for $t \geqslant t^{\tilde{\theta}}$ up to next interaction and so on; see Figure 5.5.

Definition 5.3.3. *Fix a junction J and let $(\rho_{1,0}, \ldots, \rho_{4,0})$ be an equilibrium at J. We say that road I_i is an active constraint if $f(\rho_{i,0}) = \gamma_i^{max}(\rho_{i,0})$. This precisely means that the constraint $\hat{\gamma}_i = \gamma_i^{max}(\rho_{i,0})$ is an active constraint for the LP problem (5.2.15).*

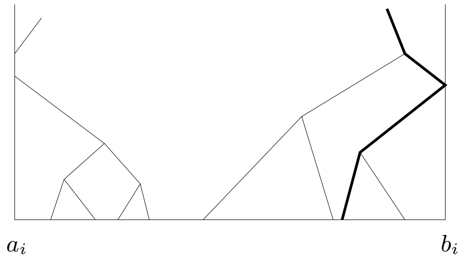


Fig. 5.5. The trace of a wave y^θ on a road.

Using the results in the appendix, we can bound the number of waves along a wave-front tracking approximate solution for the Riemann solver described in Section 5.2 and a network formed by a single junction J. More precisely, we can prove condition (H*) of Section 4.3.1. Then applying Theorem 4.3.9, the bounds on number of waves and interactions are granted. We first need to introduce an appropriate functional.

Definition 5.3.4. *Fix a network formed by a single junction J and a wave-front tracking approximate solution ρ.*
We say that a wave θ of $\rho(0)$ is an initial datum wave. The wave θ remains an initial datum wave if it interacts only with waves of the initial datum (and not with the junction). We indicate by $ID(t)$ the number of initial datum waves in $\rho(t)$.
For every $t > 0$, we say that a wave θ of $\rho(t)$ is a GG–wave if:

(GG.1) θ is not an initial datum wave,
(GG.2) θ is on an outgoing road,
(GG.3) $\rho_-^\theta(t)$ and $\rho_+^\theta(t)$ are good data,
(GG.4) $f(\rho_-^\theta(t)) > f(\rho_+^\theta(t))$.

We indicate by $GG(t)$ the number of GG–waves in $\rho(t)$.
For every $t > 0$, we say that a wave θ of $\rho(t)$ is a BS–wave if:

(BS.1) θ is not an initial datum wave,
(BS.2) θ is on an outgoing road,
(BS.3) $\rho_-^\theta(t)$ is a good datum and $\rho_+^\theta(t)$ is a bad datum,
(BS.4) $f(\rho_-^\theta(t)) > f(\rho_+^\theta(t))$, so the wave has negative speed.

We indicate by $BS(t)$ the number of BS–waves in $\rho(t)$.
Finally we define the functional:

$$N(\rho(t)) = 5 \cdot ID(t) + 2 \cdot GG(t) + BS(t).$$

Remark 5.3.5. Note that every GG-wave is a rarefaction wave in an outgoing road with positive velocity, while every BS-wave is a big shock wave in an outgoing road with negative velocity.

Proposition 5.3.6. *Consider a network formed by a single junction J with two incoming and two outgoing roads and the Riemann solver RS_J defined in Section 5.2. Then (H*) of Section 4.3.1 holds true.*

Proof. By definition $N(\rho(0))$ is bounded by $5 \cdot M$, where M is the number of waves in $\rho(0)$. Then, by Proposition A.1.5, there exist a finite sequence of times $0 < t_1 \leqslant t_2 \leqslant t_{5M}$ such that i) or ii) of Proposition A.1.5 occurs.

We claim that on each interval $]t_i, t_{i+1}[$ (letting $t_{5M+1} = +\infty$) there are at most two interactions with the junction, thus at most 8 waves are produced by the junction. By definition, the only types of interactions which may happen on such interval are:

a) two waves, not of initial datum, interact on a road,
b) a wave, not of initial datum or of type BS, interacts with the junction.

Clearly at each interaction of type a) the number of waves decreases. An interaction of type b) may occur only if a big shock with positive velocity interacts from an incoming road, say I_1. In this case, at most 4 waves are produced on the various roads. If no other interaction occurs, we are finished, otherwise there is a big shock interacting from road I_2 and producing at most 4 waves. If this happens, either a big shock with negative velocity or no wave is generated on road I_1. Since i) does not happen, the big shock on road I_1 has no more interactions, thus no other wave interacts with the junction.

Then the number of waves produced by the junction is bounded by $4 \cdot 5M + 8 \cdot (5M+1)$ and the number of interactions with the junction is bounded by $5M + 2 \cdot (5M+1)$. □

5.3.2 Estimates on Flux Total Variation

From now on we fix an approximate wave front tracking solution ρ, defined on the road network.

The following fundamental lemma holds. For a proof see the appendix of this Chapter.

Lemma 5.3.7. *Consider a road network $(\mathcal{I}, \mathcal{J})$. For some $K > 0$, we have*

$$Tot.Var.(f(\rho(t+,\cdot))) \leqslant e^{Kt}\, Tot.Var.(f(\rho(0+,\cdot)))$$

for each $t \geqslant 0$.

Definition 5.3.8. *Consider a road network $(\mathcal{I}, \mathcal{J})$ and consider an approximate wave front tracking solution ρ. For every road I_i, we define two curves $Y_-^{i,\rho}(t)$, $Y_+^{i,\rho}(t)$, called Boundary of External Flux, briefly BEF, in the following way. We set the initial condition $Y_-^{i,\rho}(0) = a_i$, $Y_+^{i,\rho}(0) = b_i$ (if $a_i = -\infty$, then $Y_-^{i,\rho} \equiv -\infty$ and if $b_i = +\infty$, then $Y_+^{i,\rho} \equiv +\infty$). We let $Y_\pm^{i,\rho}(t)$ follow the generalized characteristic as defined in [40], letting $Y_-^{i,\rho}(t) = a_i$ (resp. $Y_+^{i,\rho}(t) = b_i$) if the generalized characteristic reaches the boundary and*

$f'(\rho(t,a_i)) < 0$ (resp. $f'(\rho(t,b_i)) > 0$). (In this way $Y^{i,\rho}_{\pm}(t)$ may coincide with a_i or b_i for some time intervals). Let $\bar{t}$ be the first time $\bar{t}$ such that $Y^{i,\rho}_{-}(\bar{t}) = Y^{i,\rho}_{+}(\bar{t})$ (possibly $\bar{t} = +\infty$), then we let $Y^{i,\rho}_{\pm}$ be defined on $[0,\bar{t}]$. Finally, we define the sets

$$D^i_1(\rho) = \left\{(t,x) : t \in [0,\bar{t}) : Y^{i,\rho}_{-}(t) \leqslant x \leqslant Y^{i,\rho}_{+}(t)\right\},$$

and

$$D^i_2(\rho) = [0,+\infty) \times [a_i,b_i] \setminus D^i_1(\rho).$$

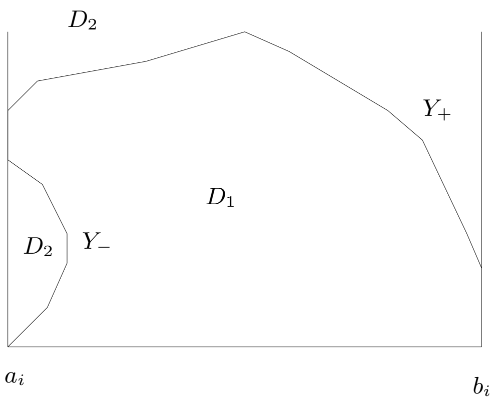


Fig. 5.6. the curves of boundary of external flux.

Clearly $Y^i_{\pm}(t)$ bound the set on which the datum is not influenced by the other roads through the junctions; see Figure 5.6.

Lemma 5.3.9. *For every $t \geqslant 0$, there exist at most two big waves on*

$$\left\{x : (t,x) \in D^i_2(\rho)\right\} \subseteq [a_i,b_i].$$

Proof. A big wave can originate at time t on road I_i from J only if road I_i has a bad datum at J at time t. If this happens, then road I_i has not a bad datum at J up to the time in which a big wave is absorbed from I_i. Then we reach the conclusion. □

We are able to state and prove the main result.

Theorem 5.3.10. *Fix a road network $(\mathcal{I},\mathcal{J})$. Given $C > 0$ and $T > 0$, there exists an admissible solution defined on $[0,T]$ for every initial data $\bar{\rho} \in cl\{\rho : TV(\rho) \leqslant C\}$, where cl indicates the closure in L^1_{loc}.*

Proof. We fix a sequence of initial data $\bar{\rho}_\nu$ piecewise constant such that $TV(\bar{\rho}_\nu) \leqslant C$ for every $\nu \geqslant 0$ and $\bar{\rho}_\nu \to \bar{\rho}$ in L^1_{loc} as $\nu \to +\infty$. For each $\bar{\rho}_\nu$ we consider an approximate wave front tracking solution ρ_ν such that $\rho_\nu(0,x) = \bar{\rho}_\nu(x)$ and rarefactions are split in rarefaction shocks of size $\frac{1}{\nu}$.

For every road I_i, we notice that on $D_1^i(\rho_\nu)$, ρ_ν is not influenced by other roads and so the estimates of [19] hold. Since the curves $Y_\pm^{i,\rho_\nu}$ are uniformly Lipschitz continuous, they converge, up to a subsequence, to a limit curve and hence the regions $D_1^i(\rho_\nu)$ "converge" to a limit region D_1^i. Then $\rho_\nu \to \rho$ in L^1_{loc} on D_1^i with ρ admissible solution to the Cauchy problem.

On $D_2^i := [0,+\infty[\times[a_i,b_i]\backslash D_1^i$, we have that, up to a subsequence, $\rho_\nu \rightharpoonup^* \rho$ weak* on L^1 and, using Lemma 5.3.7 and Theorem 2.5.4 $f(\rho_\nu) \to \bar{f}$ in L^1 for some $\bar{f}$. By Lemma 5.3.9, there are at most two big waves on D_2^i for every time, hence, splitting the domain D_2^i in a finite number of pieces where we can invert the function f, getting $\rho_\nu \to f^{-1}(\bar{f})$ in L^1. Together with $\rho_\nu \rightharpoonup^* \rho$ weak* on L^1, we conclude that $\rho_\nu \to \rho$ strongly in L^1.

The other requirements of the definition of admissible solution are clearly satisfied. □

5.4 Lipschitz Continuous Dependence

In this section we again assume (H1) and for every junctions we follow the notation (5.3.23). We present a counterexample to the Lipschitz continuous dependence by initial data with respect to the L^1-norm. The continuous dependence by initial data with respect the L^1-norm remains an open problem. The counterexample is constructed using shifts of waves as in the spirit of Section 2.7, to which we refer the reader for general theory.

We show that, for every $C > 0$, it is possible to choose two piecewise constant initial data, which are exactly the same except for a shift ξ of a discontinuity, such that the L^1–distance of the two corresponding solutions increases by the multiplicative factor C. Obviously, the L^1–distance of the initial data is finite and given by $|\xi\, \Delta\rho|$, where ξ is the shift and $\Delta\rho$ is the jump across the corresponding discontinuity. From now on, we consider a junction J, a distributional matrix A satisfying condition (C) in the form (5.3.23), with I_1, I_2 as incoming roads and I_3, I_4 as outgoing ones. Moreover we assume $\alpha < \beta$.

First we need some technical lemmas.

Lemma 5.4.1. *Let us consider a junction J with incoming roads I_1 and I_2 and outgoing roads I_3 and I_4. If a wave on a road I_i ($i \in \{1,\dots,4\}$) interacts with J without producing waves in the same road I_i and if ξ_i is the shift of the wave in I_i, then the shift ξ_j produced in a different road I_j ($j \in \{1,\dots,4\}\backslash\{i\}$) satisfies:*

$$\xi_j\left(\rho_j^+ - \rho_j^-\right) = \frac{\Delta\gamma_j}{\Delta\gamma_i}\xi_i\left(\rho_i^+ - \rho_i^-\right), \tag{5.4.24}$$

where $\Delta\gamma_l$ ($l \in \{i, j\}$) represents the variation of the flux in the road I_l and ρ_l^-, ρ_l^+ ($l \in \{i, j\}$) are the states at J in the road I_l respectively before and after the interaction.

Proof. For simplicity let us consider the case $i = 1$ and $j = 3$, the other cases being completely similar. Applying the shift ξ_1 to the wave (ρ_1^+, ρ_1^-), the interaction of the wave with J is shifted in time by

$$-\xi_1 \frac{\rho_1^+ - \rho_1^-}{f(\rho_1^+) - f(\rho_1^-)} = -\xi_1 \frac{\rho_1^+ - \rho_1^-}{\Delta\gamma_1}.$$

The shift in time in I_3 must be the same and so

$$\xi_1 \frac{\rho_1^+ - \rho_1^-}{\Delta\gamma_1} = \xi_3 \frac{\rho_3^+ - \rho_3^-}{\Delta\gamma_3},$$

which concludes the lemma. □

Remark 5.4.2. It is easy to understand that the coefficient of multiplication $\Delta\gamma_j/\Delta\gamma_i$ in the previous Lemma depends by the entries of the matrix A. For example, under the same hypotheses of the previous lemma, if a wave in the I_1 road interacts with J producing a variation of the flux $\Delta\gamma_1$ and if no wave is produced in I_1 and I_2, then

$$\Delta\gamma_3 = \alpha\Delta\gamma_1, \qquad \Delta\gamma_4 = (1 - \alpha)\Delta\gamma_1.$$

Consequently in this case

$$\frac{\Delta\gamma_3}{\Delta\gamma_1} = \alpha, \qquad \frac{\Delta\gamma_4}{\Delta\gamma_1} = 1 - \alpha.$$

The following lemma is the first step in order to show that the Lipschitz dependence by initial data does not hold in our setting. More precisely, we show that there exists a simple configuration of waves and of shifts, which, after some interactions with J, produces an increase of the L^1-distance, going to a similar configuration.

Lemma 5.4.3. *There exists an initial datum given by $(\rho_{1,0}, \rho_{2,0}, \rho_{3,0}, \rho_{4,0})$, that is an equilibrium configuration at J, a wave $(\bar\rho_2, \rho_{2,0})$ on road I_2, waves $(\rho_{3,0}, \rho_3^*)$ with shift $\xi_{3,0}$ and $(\rho_3^*, \bar\rho_3)$ on road I_3 such that the followings happen in chronological order:*

1. *the initial distance in L^1 is $\xi_{3,0} |\rho_{3,0} - \rho_3^*|$;*
2. *the wave $(\rho_{3,0}, \rho_3^*)$ in I_3 with shift $\xi_{3,0}$ interacts with J;*
3. *waves are produced only in I_2 and I_4;*
4. *the wave on road I_2 interacts with $(\bar\rho_2, \rho_{2,0})$ producing a new wave;*
5. *the new wave from road I_2 interacts with J;*
6. *waves are produced only in I_3 and I_4;*

7. the new wave on I_3 interacts with $(\rho_3^*, \bar{\rho}_3)$;
8. in I_4 the L^1–distance after the interactions, is equal to

$$2\frac{1-\beta}{\beta}\left|\xi_{3,0}\left(\rho_3^* - \rho_{3,0}\right)\right|,$$

and the L^1–distance on road I_3 is equal to $\xi_{3,0}\left|\rho_{3,0} - \rho_3^*\right|$.

Proof. Let $(\rho_{1,0}, \rho_{2,0}, \rho_{3,0}, \rho_{4,0})$ be an equilibrium configuration in J such that

$$0 < \rho_{1,0} < \sigma, \quad 0 < \rho_{2,0} < \sigma, \quad 0 < \rho_{3,0} < \sigma, \quad 0 < \rho_{4,0} < \sigma.$$

In road I_3, we consider a wave with negative speed $(\rho_{3,0}, \rho_3^*)$ with shift $\xi_{3,0}$. Since $(\rho_{3,0}, \rho_3^*)$ has negative speed, then $\rho_3^* > \tau(\rho_{3,0})$. Initially the L^1–distance of the two solutions is given by $|\xi_{3,0}(\rho_{3,0} - \rho_3^*)|$. When this wave interacts with J, new waves are produced in I_2 and I_4. It is possible, since $\alpha < \beta$. Therefore the new solution to the Riemann problem at J is given by

$$\left(\rho_{1,0}, \hat{\rho}_2, \hat{\rho}_3, \hat{\rho}_4\right),$$

where $\tau(\rho_{2,0}) < \hat{\rho}_2 < 1$, $0 < \hat{\rho}_4 < \rho_{4,0}$. Moreover some shifts $\hat{\xi}_2$ and $\hat{\xi}_4$ are produced in roads I_2 and I_4 respectively, where obviously $\hat{\xi}_2$ has the same sign of $\xi_{3,0}$ while $\hat{\xi}_4$ has opposite sign. By Lemma 5.4.1, we have

$$\begin{cases} \hat{\xi}_2(\hat{\rho}_2 - \rho_{2,0}) = \frac{1}{\beta}\xi_{3,0}(\rho_3^* - \rho_{3,0}), \\ \hat{\xi}_4(\hat{\rho}_4 - \rho_{4,0}) = \frac{1-\beta}{\beta}\xi_{3,0}(\rho_3^* - \rho_{3,0}). \end{cases}$$

If $0 < \bar{\rho}_2 < \tau(\hat{\rho}_2)$, then the wave $(\bar{\rho}_2, \rho_{2,0})$ in the road I_2 with shift $\bar{\xi}_2 = 0$ interacts with the wave $(\rho_{2,0}, \hat{\rho}_2)$ producing a wave $(\bar{\rho}_2, \hat{\rho}_2)$ with positive speed and with shift $\tilde{\xi}_2$. In this case:

$$\tilde{\xi}_2(\hat{\rho}_2 - \bar{\rho}_2) = \hat{\xi}_2(\hat{\rho}_2 - \rho_{2,0}) = \frac{1}{\beta}\xi_{3,0}(\rho_3^* - \rho_{3,0}).$$

Then, after the interaction of the wave $(\bar{\rho}_2, \hat{\rho}_2)$ with J, the new solution of the Riemann problem at J is given by

$$\left(\rho_{1,0}, \bar{\rho}_2, \hat{\rho}_3, \bar{\rho}_4\right),$$

where $0 < \hat{\rho}_3 < \tau(\rho_3^*)$ and $0 < \bar{\rho}_4 < \hat{\rho}_4$. So in the roads I_3 and I_4 new shifts $\hat{\xi}_3$ and $\bar{\xi}_4$ are created, where:

$$\begin{cases} \hat{\xi}_3(\rho_3^* - \hat{\rho}_3) = \beta\tilde{\xi}_2(\hat{\rho}_2 - \bar{\rho}_2) = \xi_{3,0}(\rho_3^* - \rho_{3,0}), \\ \bar{\xi}_4(\hat{\rho}_4 - \bar{\rho}_4) = (1-\beta)\tilde{\xi}_2(\hat{\rho}_2 - \bar{\rho}_2) = \frac{1-\beta}{\beta}\xi_{3,0}(\rho_3^* - \rho_{3,0}). \end{cases}$$

Now, if $\tau(\hat{\rho}_3) < \bar{\rho}_3 < 1$, then the wave $(\rho_3^*, \bar{\rho}_3)$ with shift $\bar{\xi}_3 = 0$ interacts in I_3 with the wave $(\hat{\rho}_3, \rho_3^*)$ producing a wave $(\hat{\rho}_3, \bar{\rho}_3)$ with negative speed and with shift $\tilde{\xi}_3$ such that

$$\tilde{\xi}_3(\bar{\rho}_3 - \hat{\rho}_3) = \hat{\xi}_3(\rho_3^* - \hat{\rho}_3) = \xi_{3,0}(\rho_3^* - \rho_{3,0}).$$

If the two waves on road I_4 do non interact, and this happens choosing appropriately the position of waves, then in the road I_4 the L^1–distance is

$$2\frac{1-\beta}{\beta}\left|\xi_{3,0}(\rho_3^* - \rho_{3,0})\right|,$$

and so the lemma is proved. □

Applying repeatedly Lemma 5.4.3, we produce a counterexample to the Lipschitz continuous dependence by initial data as the next proposition shows.

Proposition 5.4.4. *Let $C > 0$, J be a junction and let $(\rho_{1,0}, \ldots, \rho_{4,0})$ be an equilibrium configuration as in Lemma 5.4.3. There exist two piecewise constant initial data satisfying the equilibrium configuration at J such that the L^1-distance between the corresponding two solutions increases by the multiplication factor C.*

Proof. Let n be big enough so that

$$\left(1 + 2n\frac{1-\beta}{\beta}\right) > C.$$

We want to define an initial data that provides the desired increase. We choose ρ_3^* and two finite sequences $(\bar{\rho}_2^i)$, $(\bar{\rho}_3^i)$, $i = 1, \ldots, n$, so that, letting $\hat{\rho}_2^i$, $\hat{\rho}_3^i$ be the states determined as in Lemma 5.4.3, we have:

$$\begin{cases} \rho_3^* \in]\tau(\rho_{3,0}), 1], & \\ \bar{\rho}_2^i \in [0, \tau(\hat{\rho}_2^i)[, & i = 1, \ldots, n, \\ \bar{\rho}_3^i \in]\tau(\hat{\rho}_3^i), 1], & i = 1, \ldots, n. \end{cases}$$

It is easy to check that these sequences can be defined by induction.

The piecewise constant initial data in I_3 is given by

$$\begin{cases} \rho_{3,0}, & \text{if } 0 < x < x^*, \\ \rho_3^*, & \text{if } x^* < x < \hat{x}_1, \\ \hat{\rho}_3^1, & \text{if } \hat{x}_1 < x < \hat{x}_2, \\ \vdots & \ldots \\ \bar{\rho}_3^n, & \text{if } \tilde{x}_n < x, \end{cases}$$

where the values x^*, $\hat{x}_1, \ldots, \hat{x}_n$ are to be determined in the sequel. If $\xi_{3,0}$ denotes the shift of the wave $(\rho_{3,0}, \rho_3^*)$ and if no more shifts are present, then the L^1–distance of initial data is given by

$$|\xi_{3,0}|\left(\rho_3^* - \rho_{3,0}\right).$$

The initial data on I_2 is

$$\begin{cases} \rho_{2,0}, & \text{if } \tilde{x}_1 < x < 0, \\ \hat{\rho}_2^1, & \text{if } \tilde{x}_2 < x < \tilde{x}_1, \\ \vdots & \dots \\ \hat{\rho}_2^n, & \text{if } x < \tilde{x}_n \\ \vdots & \dots, \end{cases}$$

where $\tilde{x}_1, \dots, \tilde{x}_n$ are to be chosen appropriately.

The speed of the wave $(\rho_{3,0}, \rho_3^*)$ is given by the Rankine-Hugoniot condition

$$\frac{f(\rho_{3,0}) - f(\rho_3^*)}{\rho_{3,0} - \rho_3^*},$$

and consequently the time needed to go to the junction J is

$$\bar{T} = -\frac{(\rho_{3,0} - \rho_3^*)\, x^*}{f(\rho_{3,0}) - f(\rho_3^*)}.$$

Clearly we adjust $\bar{T}$, choosing x^*. Applying n times Lemma 5.4.3 and adjusting the interaction times by choosing appropriately $\bar{x}_i$, $\tilde{x}_i$, $i \in \{1, \dots, n\}$, we can create $2n$ waves on road I_4 that do not interact together before the end of these n cycles and so we deduce that, at the end, the L^1–distance of the two solutions is given by

$$\left(1 + 2n\frac{1-\beta}{\beta}\right) |\xi_{3,0}(\rho_3^* - \rho_{3,0})|\,,$$

which concludes the proof. □

Remark 5.4.5. The process described in the proof of Proposition 5.4.4 cannot be infinitely repeated. In fact, the sequences $\bar{\rho}_2^i$, $\bar{\rho}_3^i$ are monotonic and so $\bar{\rho}_3^{i+1} - \bar{\rho}_3^i \sim \frac{\bar{\rho}_3^1}{n}$ as n goes to infinity. Then the corresponding shifts on I_3 tend to infinity, letting waves interact with each other on road I_4. Therefore, with this method, it is not possible to produce a blow-up of the L^1-distance in finite time.

5.5 Time Dependent Traffic

Let us consider a model of traffic including traffic lights and time dependent traffic. The latter means that the choice of drivers at junctions may depend on the period of the day, for instance during the morning the traffic flows towards some specific parts of the network and during the evening it may flow back. This means that the matrix A may depend on time t.

Consider a single junction J with two incoming roads I_1, I_2 and two outgoing ones I_3 and I_4. Let $\alpha = \alpha(t)$, $\beta = \beta(t)$ be two piecewise constant functions such that

$$0 < \alpha(t) < 1, \qquad 0 < \beta(t) < 1, \qquad \alpha(t) \neq \beta(t), \tag{5.5.25}$$

for each $t \geqslant 0$. Moreover let $\chi_1 = \chi_1(t), \chi_2 = \chi_2(t)$ be piecewise constant maps such that

$$\chi_1(t) + \chi_2(t) = 1, \qquad \chi_i(t) \in \{0, 1\}, \qquad i = 1, 2,$$

for each $t \geqslant 0$. The two maps represent traffic lights, the value 0 corresponding to red light and the value 1 to green light.

Definition 5.5.1. *Consider $\rho = (\rho_1, \ldots, \rho_4)$ with bounded variation. We say that ρ is an admissible solution at the junction J if*

1. *ρ is a weak solution at the junction J;*
2. *$f(\rho_3(t, a_3+)) = \alpha(t)\chi_1(t) f(\rho_1(t, b_1-)) + \beta(t)\chi_2(t) f(\rho_2(t, b_2-))$ for almost every $t > 0$;*
3. *$f(\rho_4(t, a_4+)) = (1 - \alpha(t))\chi_1(t) f(\rho_1(t, b_1-)) + (1 - \beta(t))\chi_2(t) f(\rho_2(t, b_2-))$ for almost every $t > 0$;*
4. *$f(\rho_1(t, b_1-)) + f(\rho_2(t, b_2-))$ is maximum for almost every $t > 0$.*

The construction of the solution can be done as in Section 5.3. However, the total variation of $f(\rho)$ does not depend continuously on the total variation of the maps $\alpha(\cdot)$, $\beta(\cdot)$. Indeed, let us suppose that there are no traffic lights, i.e. $\chi_i \equiv 1$, and let

$$\alpha(t) = \begin{cases} \eta_1, & \text{if } 0 \leqslant t \leqslant \bar{t}, \\ \eta_2, & \text{if } \bar{t} \leqslant t \leqslant T, \end{cases} \qquad \beta(t) = \begin{cases} \eta_2, & \text{if } 0 \leqslant t \leqslant \bar{t}, \\ \eta_1, & \text{if } \bar{t} \leqslant t \leqslant T, \end{cases}$$

where $0 < \eta_2 < \eta_1 < \dfrac{1}{2}$ and $0 < \bar{t} < T$. Consider $(\rho_{1,0}, \ \rho_{2,0}, \ \rho_{3,0}, \ \rho_{4,0})$, as initial data where

$$f(\rho_{1,0}) = f(\rho_{4,0}) = f(\sigma), \qquad f(\rho_{2,0}) = f(\rho_{3,0}) = \frac{\eta_1}{1 - \eta_2} f(\sigma),$$

and

$$\sigma < \rho_{2,0} < 1, \qquad 0 < \rho_{3,0} < \sigma.$$

This is an equilibrium configuration and hence the solution of the Riemann problem for $0 \leqslant t \leqslant \bar{t}$. At time $t = \bar{t}$ we have to solve a new Riemann problem. Let $(\hat{\rho}_1, \ \hat{\rho}_2, \ \hat{\rho}_3, \ \hat{\rho}_4)$ be the new solution. We have:

$$f(\hat{\rho}_2) = f(\hat{\rho}_4) = f(\sigma), \qquad f(\hat{\rho}_1) = f(\hat{\rho}_3) = \frac{\eta_1}{1 - \eta_2} f(\sigma).$$

Now, if $\eta_1 \to \eta_2$, then

$$\text{Tot.Var.}\big(\alpha; [0, T]\big) \longrightarrow 0, \qquad \text{Tot.Var.}\big(\beta; [0, T]\big) \longrightarrow 0,$$

but

$$(f(\rho_{1,0}),\ f(\rho_{2,0})) \to \left(f(\sigma),\ \frac{\eta_2}{1-\eta_2} f(\sigma)\right),$$

and

$$(f(\hat\rho_1),\ f(\hat\rho_2)) \to \left(\frac{\eta_2}{1-\eta_2} f(\sigma),\ f(\sigma)\right),$$

hence $\text{Tot.Var.}\big(f(\rho); [0,T]\big)$ is bounded away from zero.

5.6 Total Variation of the Fluxes

Let J be a junction with 3 incoming roads and 3 outgoing ones. We show, with an example, that the total variation of the flux may increase if a wave arrives to J from an incoming road. Let us suppose that the matrix A is given by

$$A = \begin{pmatrix} \frac{1}{2}-\varepsilon & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{2} & \frac{1}{2}+\varepsilon \\ \frac{1}{6}+\varepsilon & 0 & \frac{1}{6}-\varepsilon \end{pmatrix}, \tag{5.6.26}$$

with $\varepsilon > 0$. Notice that the matrix A satisfies condition (C) for every $\varepsilon > 0$ small enough.

Let us choose $\rho_1,\ \rho_{1,0}, ..., \ \rho_{6,0} \in [0,1]$ such that

$$\rho_{1,0} = \rho_{4,0} = \rho_{5,0} = \sigma, \qquad \sigma < \rho_{2,0} < 1, \qquad \sigma < \rho_{3,0} < 1,$$
$$0 < \rho_{6,0} < \sigma, \qquad 0 < \rho_1 < \sigma,$$
$$f(\rho_{2,0}) = \frac{1+36\varepsilon+36\varepsilon^2}{3(1+6\varepsilon)},\ f(\rho_{3,0}) = \frac{1-6\varepsilon}{1+6\varepsilon},\ f(\rho_{6,0}) = \frac{1}{6}+\varepsilon+\frac{(1-6\varepsilon)^2}{6(1+6\varepsilon)}.$$

Assuming that $f(\sigma) = 1$, $(\rho_{1,0}, ..., \ \rho_{6,0})$ is an equilibrium configuration and ρ given by

$$\rho_1(0,x) = \begin{cases} \rho_{1,0}, & \text{if } x_1 \leqslant x \leqslant b_1, \\ \rho_1, & \text{if } x < x_1, \end{cases} \qquad \rho_i(0,\cdot) \equiv \rho_{i,0}, \quad i = 2, ..., 6,$$

is a solution (see Figure 5.7). Moreover the point $\big(f(\rho_{1,0}), f(\rho_{2,0}), f(\rho_{3,0})\big)$ is given by the intersection of the planes

$$\left(\frac{1}{2}-\varepsilon\right)\gamma_1 + \frac{1}{2}\gamma_2 + \frac{1}{3}\gamma_3 = 1, \qquad \frac{1}{3}\gamma_1 + \frac{1}{2}\gamma_2 + \left(\frac{1}{2}+\varepsilon\right)\gamma_3 = 1, \qquad \gamma_1 = 1.$$

At some time, say $\bar t$, the wave $(\rho_1, \rho_{1,0})$ interacts with the junction. Let $(\hat\rho_1, ..., \hat\rho_6)$ be the solution of the Riemann problem at the junction for the data $(\rho_1,\ \rho_{2,0}, ..., \ \rho_{6,0})$. If $f(\rho_1)$ is sufficiently near to 1, then we have:

$$f(\hat\rho_1) = f(\rho_1), \qquad f(\hat\rho_2) = 2 - \tfrac{5-36\varepsilon^2}{3(1+6\varepsilon)} f(\rho_1),$$
$$f(\hat\rho_3) = \tfrac{1-6\varepsilon}{1+6\varepsilon} f(\rho_1), \quad f(\hat\rho_4) = f(\hat\rho_5) = 1,$$
$$f(\hat\rho_6) = \tfrac{1+36\varepsilon^2}{3(1+6\varepsilon)} f(\rho_1).$$

Therefore

$$\text{Tot.Var.}\big(f(\rho(\bar t-,\cdot))\big) = 1 - f(\rho_1),$$

and

$$\text{Tot.Var.}\big(f(\rho(\bar t+,\cdot))\big) = \frac{3(1-2\varepsilon)}{1+6\varepsilon}\big(1 - f(\rho_1)\big) > 2\,\text{Tot.Var.}\big(f(\rho(\bar t-,\cdot))\big).$$

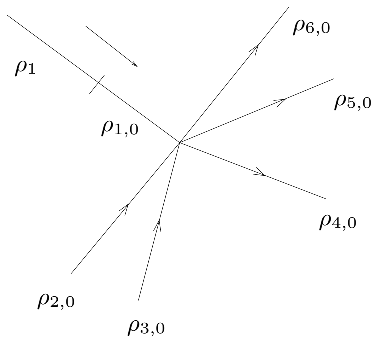


Fig. 5.7. Configuration at J.

5.7 Total Variation of the Densities

Consider a junction J with two incoming roads and two outgoing ones that we parameterize with the intervals $]-\infty, b_1]$, $]-\infty, b_2]$, $[a_3, +\infty[$, $[a_4, +\infty[$ respectively. We suppose that $0 < \beta < \alpha < 1/2$, where α and β are the entries of the matrix A as in (5.3.23).

Define a solution ρ by

$$\rho_1(0,x) = \begin{cases} \rho_{1,0}, & \text{if } x_1 \leqslant x \leqslant b_1, \\ \rho_1, & \text{if } x < x_1, \end{cases} \qquad \rho_2(0,x) = \rho_{2,0},$$
$$\rho_3(0,x) = \rho_{3,0}, \quad \rho_4(0,x) = \rho_{4,0}, \tag{5.7.27}$$

where ρ_1, $\rho_{1,0}$, $\rho_{2,0}$, $\rho_{3,0}$, $\rho_{4,0}$ are constants such that

$$\sigma < \rho_{2,0} < 1, \quad \sigma < \rho_{3,0} < 1, \quad 0 \leqslant \rho_1 < \sigma, \quad \rho_{1,0} = \rho_{4,0} = \sigma, \tag{5.7.28}$$

$$f(\rho_{1,0}) = f(\rho_{4,0}) = f(\sigma), \qquad f(\rho_{2,0}) = f(\rho_{3,0}) = \frac{\alpha}{1-\beta} f(\sigma),$$

so $(\rho_{1,0},\ \rho_{2,0},\ \rho_{3,0},\ \rho_{4,0})$ is an equilibrium configuration.

After some time the wave $(\rho_1, \rho_{1,0})$ interacts with the junction. Denote with $(\hat\rho_1, \hat\rho_2, \hat\rho_3, \hat\rho_4)$ the solution to the Riemann problem at the junction when the initial datum is $(\rho_1, \rho_{2,0}, \rho_{3,0}, \rho_{4,0})$; see Figure 5.8. By (5.7.27) and (5.7.28),

$$f(\hat\rho_1) = f(\rho_1), \qquad f(\hat\rho_2) = \frac{f(\sigma) - (1-\alpha) f(\rho_1)}{1-\beta},$$

$$f(\hat\rho_3) = \frac{\alpha - \beta}{1-\beta} f(\rho_1) + \frac{\beta}{1-\beta} f(\sigma), \qquad f(\hat\rho_4) = f(\sigma),$$

and

$$0 < \hat\rho_3 < \sigma \leqslant \hat\rho_2 < 1. \tag{5.7.29}$$

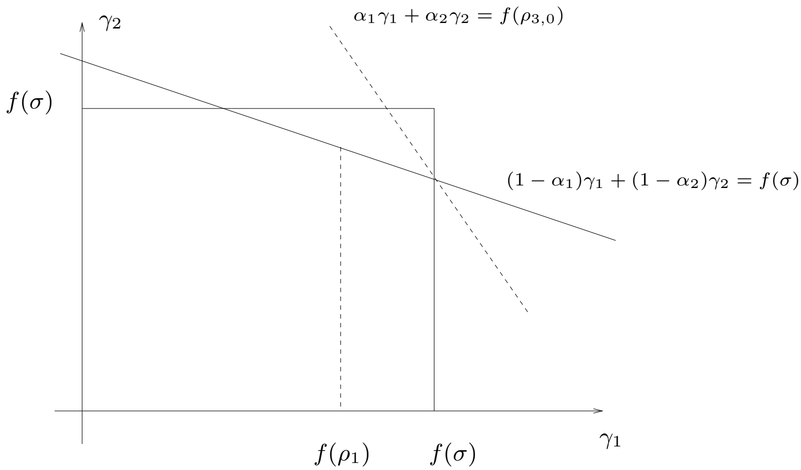


Fig. 5.8. Solution to the Riemann problem at J.

Therefore, if $\rho_1 \to \rho_{1,0} = \sigma$, then

$$f(\hat\rho_3) \longrightarrow \frac{\alpha}{1-\beta} f(\sigma) = f(\rho_{3,0}),$$

and, by (5.7.29), (5.7.28), we have $\hat\rho_3 \to \tau(\rho_{3,0})$. Then, we are able to create on the third road a wave with strength bounded away from zero using an arbitrarily small wave on the first one.

5.8 Exercises

Exercise 5.8.1. Consider a network composed by a single junction J with three incoming roads I_1, I_2 and I_3 and one outgoing road I_4. Assume that on each road the flux is given by $f(\rho) = \rho(1-\rho)$ and fix the precedence parameters $q_1 = 1$ and $q_2 = 2$. Find the solution to the Riemann problem when the initial data are given by

$$\rho_{1,0} = \frac{1}{2}, \qquad \rho_{2,0} = \frac{1}{4}, \qquad \rho_{3,0} = \frac{3}{4}, \qquad \rho_{4,0} = \frac{2}{3}.$$

Exercise 5.8.2. Consider a network composed by a single junction J with one incoming road I_1 and two outgoing roads I_2 and I_3. On each road consider the Greenberg Model, see (3.1.4), and fix the matrix

$$A = \begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \end{pmatrix}.$$

Find the solution to the Riemann problem when the initial data are given by

$$\rho_{1,0} = \frac{1}{2}, \qquad \rho_{2,0} = \frac{1}{4}, \qquad \rho_{3,0} = \frac{3}{4}.$$

Exercise 5.8.3. Discuss the possibility of defining Riemann solvers at junctions for the Underwood model (3.1.5) and the California model (3.1.7) using rules (A) and (B).

Exercise 5.8.4. Consider a road network composed by two junctions J_1 and J_2. The junction J_1 has two incoming roads I_1 and I_2 and one outgoing road I_3. The precedence coefficient is given by $q = \frac{1}{3}$. The junction J_2 has one incoming road I_3 and two outgoing roads I_4 and I_5. The matrix A for J_2 is given by

$$A = \begin{pmatrix} \frac{1}{4} \\ \frac{3}{4} \end{pmatrix}.$$

Assume that for all the roads the flux f is equal to $\rho(1-\rho)$. Consider the initial datum, constant on each road:

$$\rho_{0,1} = \rho_{0,2} = \rho_{0,4} = 0.5, \qquad \rho_{0,3} = 0.1, \qquad \rho_{0,5} = 1.$$

a) Calculate the time $T > 0$ of interaction between the waves in $I_3 = [a_3, b_3]$ generated respectively by J_1 and J_2 at time $t = 0$.
b) Compute the limit of the solution as $b_3 - a_3$ tends to zero.

Exercise 5.8.5. Consider a network formed by a single junction with two incoming roads and two outgoing ones. On each road consider the flux:

$$f(\rho) = \begin{cases} \bar{v}\rho, & \text{if } 0 \leqslant \rho \leqslant \sigma, \\ \bar{v}(2\sigma - \rho), & \text{if } \sigma \leqslant \rho \leqslant \rho_{max}. \end{cases}$$

Find an initial datum, constant on each road, and a time dependent traffic distribution matrix $A(t)$ such the the following holds. Denoting by $\rho(t)$ the corresponding solution, there exists $T > 0$ such that:

$$\text{Tot.Var.}(A(\cdot), [0, T]) < +\infty, \qquad \text{Tot.Var.}(\rho(T)) = +\infty.$$

5.9 Bibliographical Note

The idea to consider the Lighthill-Whitham-Richards on a network was proposed by Holden and Risebro [66]. They solved the Riemann problem at junctions proposing a maximization of the flux.

Existence of solution to Cauchy problems was proved in the paper by Coclite, Garavello and Piccoli [27]. The counterexample to the Lipschitz continuous dependence on initial data is contained in [27].

The point of view of Demand-Supply was introduced by Lebacque [84] and by Lebacque and Khoshyaran [85], [86].

Time dependent traffic was studied in particular by Tong Li [103], which considered fundamental diagrams varying in time for the Aw-Rascle model and its extensions.

Appendix

A.1 Technical Results

Lemma A.1.1. *Fix a junction J. Consider a trace of a wave y^θ such that*

a) y^θ is generated at time $\bar{t}$ from J,
b) y^θ interacts at time $\tilde{t} > \bar{t}$ with J,
c) y^θ does not interact with any junction on the interval $]\bar{t}, \tilde{t}[$.

Then:

i) $y^\theta(t)$ is a big shock for some $t \in]\bar{t}, \tilde{t}[$,
ii) if $y^\theta(\bar{t})$ is generated on an incoming road and $y^\theta(\tilde{t}) = (\rho_l, \rho_r)$, then ρ_l is a bad datum and $f(\rho_l) > f(\rho_r)$; for outgoing roads, ρ_r is a bad datum and it holds $f(\rho_l) < f(\rho_r)$.

Proof. Let us prove i). Assume, by contradiction, that $y^\theta(\bar{t})$ is not a big shock and y^θ does not interact with a big shock on the interval $]\bar{t}, \tilde{t}[$. Then, by c) and Lemma 4.3.6, y^θ always connects two good states. Therefore, if y^θ is on an incoming road, then its velocity is always negative, while if y^θ is on an outgoing road, then its velocity is always positive. We reach a contradiction with b).
Let us now prove ii). Again by Lemma 4.3.6, all waves produced by J on the interval $]\bar{t}, \tilde{t}[$ connect two good states. In particular, if y^θ is on an incoming road, then it always interact, on its right, with waves connecting two good states. Thus, the right state of y^θ is always good and, in particular, ρ_l is a bad datum. Since the velocity of $y^\theta(\tilde{t})$ must be positive we obtain the desired conclusion. The proof for outgoing roads is entirely similar. □

Lemma A.1.2. *Fix a junction J. Consider a trace of a wave y^θ such that*

a) y^θ is a big shock generated at time $\bar{t}$ from J,
b) y^θ interacts at time $\tilde{t} > \bar{t}$ with J,
c) y^θ interacts only with waves produced by J on the interval $]\bar{t}, \tilde{t}[$ (and not with junctions).

Then, if $y^\theta(\bar{t})$ is generated on an incoming road, y^θ must interact on the interval $]\bar{t}, \tilde{t}[$ with a wave (ρ_-, ρ_+) such that ρ_-, ρ_+ are good data for the incoming road and $f(\rho_-) < f(\rho_+)$. A similar conclusion holds for outgoing roads with the condition $f(\rho_-) > f(\rho_+)$.

Proof. By Lemma 4.3.6, all waves produced by J on the interval $]\bar{t}, \tilde{t}[$ connect two good states. Assume that y^θ is on an incoming road. Then there exists a time $t \in]\bar{t}, \tilde{t}[$ such that y^θ interacts at time t with a wave (ρ_-, ρ_+), y^θ has negative velocity in a left neighborhood of t and positive velocity in a right neighborhood of t. Then $f(\rho^\theta_-(t)) > f(\rho_-)$ and $f(\rho^\theta_-(t)) < f(\rho+)$. The case of outgoing roads is entirely similar. □

Lemma A.1.3. *Fix a junction J and let $(\rho_{1,0}, \ldots, \rho_{4,0})$ be an equilibrium at J. Then at least two roads are active constraints; moreover it holds:*

i) if $\rho_{i,0}$ is a bad datum, then I_i is an active constraint,
ii) if I_i is an active constraint, then $\rho_{i,0}$ is a bad datum or $\rho_{i,0} = \sigma$.

Proof. By condition (C), the solution to the LP problem (5.2.15) is obtained on a vertex of the region Ω. Then there are at least two active constraints. Now, if $\rho_{i,0}$ is a bad datum, then $f(\rho_{i,0}) = \gamma_i^{max}(\rho_{i,0})$.
If I_i is an active constraint, then, by definition, $f(\rho_{i,0}) = \gamma_i^{max}(\rho_{i,0})$. This equality is always true for bad data and for good data only in the case of equality with σ. □

Lemma A.1.4. *Fix a junction J and let $(\rho_{1,0}, \ldots, \rho_{4,0})$ be an equilibrium at J. Assume that a wave $(\rho_i, \rho_{i,0})$, $i \in \{1, 2\}$, interacts with J and $f(\rho_i) < f(\rho_{i,0})$. Let $\hat{\rho}_j$ ($j \in \{3, 4\}$) be the states at J in the outgoing roads after the interaction. Then $f(\hat{\rho}_j) \leqslant f(\rho_{j,0})$ for $j = 3, 4$, where the inequality is strict for at least one $j \in \{3, 4\}$.*
The same conclusion holds if a big shock $(\rho_{j,0}, \rho_j)$, $j \in \{3, 4\}$, interacts with J.

Proof. Let us start considering the case of an interacting wave $(\rho_1, \rho_{1,0})$ (the case of interaction from road I_2 being entirely similar). Notice that, by hypotheses, the wave $(\rho_1, \rho_{1,0})$ is a shock and ρ_1 is a bad datum. Hence $f(\hat{\rho}_1) \leqslant f(\rho_1) < f(\rho_{1,0})$.
Assume first that the incoming road I_2 is an active constraint; see Figure 5.9. Then $f(\hat{\rho}_i) \leqslant f(\rho_{i,0})$ $(i = 1, 2)$ and the conclusion holds.

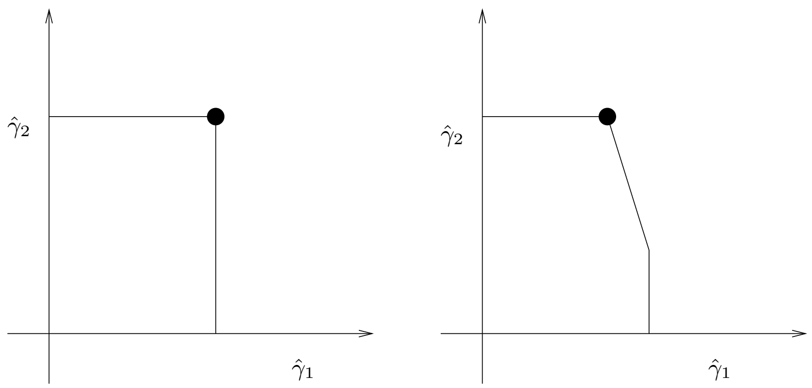


Fig. 5.9. Case of I_2 active constraint: I_1 active constraint (left), an outgoing road active constraint (right). The big dot indicates the solution to the LP problem (5.2.15).

Assume now that only one of the outgoing roads, say road I_3, is an active constraint; see Figure 5.10. Then $f(\hat{\rho}_3) \leqslant f(\rho_{3,0})$ and so

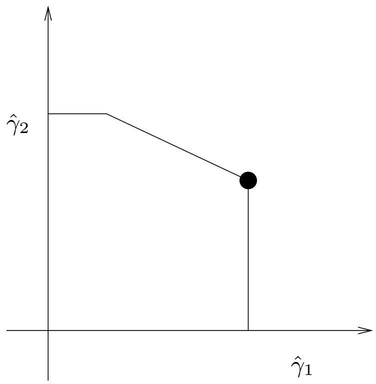


Fig. 5.10. Case of I_1 and I_3 active constraints and I_4 not active constraints. The big dot indicates the solution to the LP problem (5.2.15).

$$f(\rho_{2,0}) = -\frac{\alpha}{\beta} f(\rho_{1,0}) + \frac{f(\rho_{3,0})}{\beta},$$

$$f(\hat{\rho}_2) = -\frac{\alpha}{\beta} f(\hat{\rho}_1) + \frac{f(\hat{\rho}_3)}{\beta} \leqslant -\frac{\alpha}{\beta} f(\hat{\rho}_1) + \frac{f(\rho_{3,0})}{\beta}.$$

Since I_3 is an active constraint, and not I_4, we get:

$$\alpha < \beta,$$

then

$$\begin{aligned} f(\hat{\rho}_4) = (1-\alpha) f(\hat{\rho}_1) + (1-\beta) f(\hat{\rho}_2) &\leqslant \frac{\beta - \alpha}{\beta} f(\hat{\rho}_1) + \frac{1-\beta}{\beta} f(\rho_{3,0}) \\ &< \frac{\beta - \alpha}{\beta} f(\rho_{1,0}) + \frac{1-\beta}{\beta} f(\rho_{3,0}) = f(\rho_{4,0}), \end{aligned}$$

the last inequality being true only because $\alpha < \beta$.
Assume now that both the outgoing roads are active constraints. Then the same reasoning works considering the active constraint of the outgoing road I_3 if $\alpha < \beta$ and the active constraint of the outgoing road I_4 if $\alpha > \beta$.

We are left with the case of a big shock coming from an outgoing road, say I_3. Then, necessarily, $f(\rho_3) < f(\rho_{3,0})$, ρ_3 is a bad datum and $f(\hat{\rho}_3) < f(\rho_{3,0})$. The case of both incoming roads being active constraints leads to the same conclusion as before.
If I_3 is active constraint and not I_4, again we get easily $f(\hat{\rho}_i) \leq f(\rho_{i,0})$,

$i = 1, 2$.
Finally if I_4 is an active constraint, since the region Ω after the interaction is restricted, we conclude. □

Proposition A.1.5. *Consider a network formed by a single junction J with two incoming and two outgoing roads and the Riemann solver RS_J defined in Section 5.2. Let ρ be a wave-front tracking approximation, then the functional $N(\rho(t))$ is decreasing in time. Moreover, it decreases by at least 1 if any of the following happens:*

i) an initial datum wave interacts with the junction or with a wave in a road,
ii) a BS–wave interacts with the junction.

Proof. Fix t and assume that a BS–wave θ is generated at time t. Since by BS.1 it is not an initial datum wave, then, by BS.3 and Lemmas 4.3.6 and A.1.1, there exists θ', big shock generated at time $t' < t$, such that $y^{\theta'}$ interacts with a wave at time t producing θ. Since $y^{\theta'}$ is not a BS-wave, then it violates BS.1 or BS.4. If it violates BS.1, then the functional N decreases at least by 4. Assume therefore that it violates BS.4. It means that $y^{\theta'}$ is a big shock with positive speed and that it interacts with a wave with good-good data. This second wave could be a GG-wave and in this case N decreases at least by 1 or an initial datum wave and in this case N decreases at least by 4.

Fix t and assume now that a GG–wave θ is generated at time t. Then one of the two cases happens:

a) θ is generated by J at time t,
b) θ is generated by interaction of waves at time t on an outgoing road.

In the first case, by Lemma A.1.4, there exists a wave with increasing flux interacting with J at time t from an incoming road or a wave, which is not a big shock, interacting with J from an outgoing road at time t. If a wave with increasing flux interacts with J at time t from an incoming road, then it is necessarily a rarefaction wave with positive velocity connecting two bad data and so it is an initial datum wave. If a wave, which is not a big shock, interacts with J from an outgoing road at time t, then it is necessarily a rarefaction wave with negative velocity connecting two bad data and so it is an initial datum wave. By this interaction at most two GG–waves are generated, thus N decreases at least by 1. The case b) does not happen.

Since no initial datum wave may be generated at positive times, we conclude that $N(\rho(\cdot))$ is decreasing.

Consider now case i). If the initial datum wave interacts with another wave, then there is a decrease of at least 3. If it interacts with the junction, as in case a) above, there is a decrease of at least 1.

If ii) happens, then by Lemma A.1.4, no GG–waves are produced, then there is a decrease of at least 1. □

Lemma A.1.6. *Fix a junction J and an incoming road I_i. Let θ be a wave on road I_i, originated at time $\bar{t}$ from J with a flux decrease, i.e. $x^{\theta}(\bar{t}) = b_i$,*

$\lambda^\theta(\bar{t}) < 0$ and $f(\rho_+^\theta) < f(\rho_-^\theta)$. Let y^θ be the traced wave and assume that there exists $\tilde{t}$, the first time of interaction of y^θ with J after $\bar{t}$. Then either y^θ interacts with another junction on $]\bar{t}, \tilde{t}[$ or, letting $\theta_1, \dots, \theta_l$ be the waves interacting with y^θ at times $t_m \in]\bar{t}, \tilde{t}[$, $m = 1, \dots, l$, ($t_1 < t_2 < \dots < t_l$), we have:

$$\left|f(\rho(\tilde{t}-\varepsilon, y^\theta(\tilde{t}-\varepsilon)+)) - f(\rho(\tilde{t}-\varepsilon, y^\theta(\tilde{t}-\varepsilon)-))\right| \leqslant$$
$$\sum_{m=1}^{l} \left|f(\rho(t_m-\varepsilon, x^{\theta_m}(t_m-\varepsilon)+)) - f(\rho(t_m-\varepsilon, x^{\theta_m}(t_m-\varepsilon)-))\right|$$
$$-\left|f(\rho_-^\theta) - f(\rho_+^\theta)\right|,$$

for $\varepsilon > 0$ small enough. This means that the initial flux variation along y^θ is canceled. The same conclusion holds for an outgoing road I_j.

Proof. Consider the wave $(\rho_-^\theta, \rho_+^\theta)$ as in the statement, then it is a shock with negative velocity and $\rho_+^\theta > \max\{\rho_-^\theta, \tau(\rho_-^\theta)\}$. If y^θ interacts with another junction, then there is nothing to prove. So, we assume that y^θ does not interact with another junction. At time t_1, the wave θ_1 interacts with y^θ. We analyze first the case of interaction from the left of y^θ. We have two possibilities:

1. $\rho_-^{\theta_1} \in [0, \tau(\rho_+^\theta)]$. In this case we have total cancellation of the flux variation and so
$$\left|f(\rho_+^\theta) - f(\rho_-^{\theta_1})\right| = \left|f(\rho_-^{\theta_1}) - f(\rho_-^\theta)\right| - \left|f(\rho_-^\theta) - f(\rho_+^\theta)\right|.$$
Therefore the claim easily follows.
2. $\rho_-^{\theta_1} \in]\tau(\rho_+^\theta), \rho_+^\theta]$. In this case the wave y^θ after the time interaction t_1 is of the same type of y^θ before t_1, i.e.
$$\max\{\rho(t_1, y^\theta(t_1)-), \tau(\rho(t_1, y^\theta(t_1)-))\} < \rho(t_1, y^\theta(t_1)+).$$

We consider now the case of interaction from the right of y^θ. It is clear that $\rho_+^{\theta_1} \in]\rho_-^\theta, 1]$. If moreover $f(\rho_+^{\theta_1}) \geqslant f(\rho_-^\theta)$, then we have total cancellation of the flux and we conclude as before. If instead $f(\rho_+^{\theta_1}) < f(\rho_-^\theta)$, then the wave y^θ after the time t_1 is of the same type of y^θ before t_1.

We repeat this argument at each interaction time t_m. If at some t_m we have total cancellation of the flux, then we conclude. Therefore we may suppose that at each t_m total cancellation of the flux does not occur. Since the type of the wave y^θ does not change, we have

$$\max\{\rho(\tilde{t}-\tilde{\varepsilon}, y^\theta(\tilde{t}-\tilde{\varepsilon})-), \tau(\rho(\tilde{t}-\tilde{\varepsilon}, y^\theta(\tilde{t}-\tilde{\varepsilon})-))\} < \rho(\tilde{t}-\tilde{\varepsilon}, y^\theta(\tilde{t}-\tilde{\varepsilon})+)$$

for $\tilde{\varepsilon} > 0$ small enough and hence the speed $\lambda^\theta(\tilde{t}-\tilde{\varepsilon})$ is negative, which contradicts the fact that y^θ interacts with J at time $\tilde{t}$. □

Lemma A.1.7. *Fix a junction J and an incoming road I_i. Let θ be a wave on road I_i, originated at time $\bar{t}$ from J by a flux increase, i.e. $x^\theta(\bar{t}) = b_i$, $\lambda^\theta(\bar{t}) < 0$ and $f(\rho_+^\theta) > f(\rho_-^\theta)$. Let y^θ be the traced wave and assume that there exists $\tilde{t}$, the first time of interaction of y^θ with J after $\bar{t}$. Then y^θ interacts with other junctions in $]\bar{t}, \tilde{t}[$ or y^θ cancels the flux variation, or it produces a flux decrease at J at $\tilde{t}$, i.e.*

$$f(\rho(\tilde{t} - \varepsilon, y^\theta(\tilde{t} - \varepsilon)-)) < f(\rho(\tilde{t} - \varepsilon, y^\theta(\tilde{t} - \varepsilon)+)),$$

for $\varepsilon > 0$ small enough. The same holds for outgoing roads.

Proof. Since $\lambda^\theta(\bar{t}) < 0$ and $f(\rho_+^\theta) > f(\rho_-^\theta)$, then $\rho_-^\theta > \sigma$. Moreover the wave $(\rho_-^\theta, \rho_+^\theta)$ is a rarefaction fan, hence $\sigma < \rho_+^\theta < \rho_-^\theta$.

If an interaction on the right with a wave θ_1 happens, then $\rho_+^{\theta_1} \in]\rho_-^\theta, 1]$ and we have total cancellation of the flux variation. Therefore we may suppose that an interaction on the left with a wave θ_1 happens. In this case we have two possibilities:

1. $\rho_-^{\theta_1} \in [0, \tau(\rho_+^\theta)[$;
2. $\rho_-^{\theta_1} \in [\tau(\rho_+^\theta), \rho_+^\theta[$.

In the latter case we have total cancellation of the flux variation and so we conclude. In the first case, instead, the type of the wave changes, since

$$0 < \rho_-^{\theta_1} < \tau(\rho_+^\theta) \leqslant \sigma \leqslant \rho_+^\theta < 1.$$

The speed of the wave y^θ after this interaction is positive and if there are no more interaction, then we have the claim since $f(\rho_-^{\theta_1}) < f(\rho_+^\theta)$. Thus we suppose that an interaction with a wave θ_2 happens. If it is an interaction from the left, then the possibilities are the followings:

1. $\rho_-^{\theta_2} \in [0, \tau(\rho_+^\theta)[$. We do not have total cancellation of the flux variation, but the type of the wave does not change and the situation is identical to the previous one.
2. $\rho_-^{\theta_2} \in [\tau(\rho_+^\theta), \sigma[$. We have total cancellation of the flux variation and so we conclude.

If it is an interaction from the right, then the possibilities are the followings:

1. $\rho_+^{\theta_2} \in [\sigma, \tau(\rho_-^{\theta_1})[$. We do not have total cancellation of the flux variation, but the type of the wave does not change.
2. $\rho_+^{\theta_2} \in [\tau(\rho_-^{\theta_1}), 1]$. We have total cancellation of the flux variation and so we conclude.

The conclusion now easily follows repeating this argument. If at each interaction we do not have total cancellation of the flux variation, then we necessarily have that

$$f(\rho(\tilde{t} - \varepsilon, y^\theta(\tilde{t} - \varepsilon)-)) < f(\rho(\tilde{t} - \varepsilon, y^\theta(\tilde{t} - \varepsilon)+)),$$

for $\varepsilon > 0$ small enough, which concludes the proof. □

Lemma A.1.8. *Fix a junction J. If a wave interacts with the junction J from an incoming road at time $\bar{t}$, then*

$$Tot.Var.\big(f(\rho(\bar{t}+,\cdot))\big) = Tot.Var.\big(f(\rho(\bar{t}-,\cdot))\big). \tag{A.1.30}$$

Proof. For simplicity let us assume that I_1, I_2 are the incoming roads and I_3, I_4 are the outgoing ones. Let $(\rho_{1,0}, ..., \rho_{4,0})$ be an equilibrium configuration at the junction J. We assume that the wave is coming from the first road and that it is given by the values $(\rho_1, \rho_{1,0})$. Let us define the incoming flux

$$f^{in}(y) := \begin{cases} f(y), & \text{if } 0 \leqslant y \leqslant \sigma, \\ f(\sigma), & \text{if } \sigma \leqslant y \leqslant 1, \end{cases} \tag{A.1.31}$$

and the outgoing flux

$$f^{out}(y) := \begin{cases} f(\sigma), & \text{if } 0 \leqslant y \leqslant \sigma, \\ f(y), & \text{if } \sigma \leqslant y \leqslant 1. \end{cases} \tag{A.1.32}$$

Clearly, since the wave on the first road has positive velocity, we have

$$0 \leqslant \rho_1 < \sigma. \tag{A.1.33}$$

Let $(\hat{\rho}_1, \ldots, \hat{\rho}_4)$ be the solution to the Riemann problem at the junction J with initial data $(\rho_1, \rho_{2,0}, \rho_{3,0}, \rho_{4,0})$ (see Theorem 5.2.1). By definition of equilibrium, $(f(\rho_{1,0}), f(\rho_{2,0}))$ is the maximum point of the map E on the domain

$$\Omega_0 := \left\{(\gamma_1, \gamma_2) \in \Omega_{1,0} \times \Omega_{2,0} | A \cdot (\gamma_1, \gamma_2)^T \in \Omega_{3,0} \times \Omega_{4,0}\right\},$$

and $\big(f(\hat{\rho}_1), f(\hat{\rho}_2)\big)$ is the maximum point of the map E on the domain

$$\hat{\Omega} := \left\{(\gamma_1, \gamma_2) \in \Omega_1 \times \Omega_{2,0} | A \cdot (\gamma_1, \gamma_2)^T \in \Omega_{3,0} \times \Omega_{4,0}\right\},$$

where

$$\Omega_{j,0} \doteq \begin{cases} [0, f^{in}(\rho_{j,0})], & \text{if } j = 1, 2, \\ [0, f^{out}(\rho_{j,0})], & \text{if } j = 3, 4, \end{cases}$$

and, by (A.1.33),

$$\Omega_1 := [0, f^{in}(\rho_1)] = [0, f(\rho_1)].$$

It is also clear that

$$(f(\rho_{1,0}), f(\rho_{2,0})) \in \partial\Omega_0, \qquad (f(\hat{\rho}_1), f(\hat{\rho}_2)) \in \partial\hat{\Omega}.$$

For simplicity we use the notation (5.3.23).

We distinguish two cases. First we suppose that

$$f(\rho_1) < f(\rho_{1,0}), \tag{A.1.34}$$

(equality can not happen in the previous equation because the wave would have velocity zero). Then $\hat{\Omega} \subset \Omega_0$ and

$$f(\hat{\rho}_1) \leqslant f(\rho_1), \qquad f(\hat{\rho}_1) + f(\hat{\rho}_2) \leqslant f(\rho_{1,0}) + f(\rho_{2,0}). \tag{A.1.35}$$

We claim that

$$f(\rho_{2,0}) \leqslant f(\hat{\rho}_2), \quad f(\hat{\rho}_3) \leqslant f(\rho_{3,0}), \quad f(\hat{\rho}_4) \leqslant f(\rho_{4,0}). \tag{A.1.36}$$

The points $(f(\rho_{1,0}), f(\rho_{2,0}))$, $(f(\hat{\rho}_1), f(\hat{\rho}_2))$ are on the boundaries of Ω_0, $\hat{\Omega}$ respectively, where the function E attains the maximum, hence each one is at least on one of the curves

$$\alpha\gamma_1 + \beta\gamma_2 = f^{out}(\rho_{3,0}), \quad (1-\alpha)\gamma_1 + (1-\beta)\gamma_2 = f^{out}(\rho_{4,0}), \quad \gamma_2 = f^{in}(\rho_{2,0}).$$

Let us assume that the two points are on the same curve, the other cases being similar,

$$\alpha\gamma_1 + \beta\gamma_2 = f^{out}(\rho_{3,0}). \tag{A.1.37}$$

Observe that the map E is increasing on the curve

$$\gamma_1 \mapsto \left(\gamma_1, \frac{f^{out}(\rho_{3,0})}{\beta} - \frac{\alpha}{\beta}\gamma_1\right),$$

otherwise we contradict the maximality of E at $(f(\rho_{1,0}), f(\rho_{2,0}))$. Thus $\alpha < \beta$, $\hat{\rho}_1 = \rho_1$, the first two inequalities in (A.1.36) hold and

$$f(\hat{\rho}_1) = f(\rho_1), \quad f(\hat{\rho}_2) > f(\rho_{2,0}), \quad f(\hat{\rho}_3) = f(\rho_{3,0}) = f^{out}(\rho_{3,0}). \tag{A.1.38}$$

On the other hand, by (A.1.35), we have

$$\begin{aligned} f(\hat{\rho}_4) &= (1-\alpha)f(\hat{\rho}_1) + (1-\beta)f(\hat{\rho}_2) \\ &\leqslant (1-\alpha)\big(f(\rho_{1,0}) + f(\rho_{2,0}) - f(\hat{\rho}_2)\big) + (1-\beta)f(\hat{\rho}_2) \\ &= (1-\alpha)\big(f(\rho_{1,0}) + f(\rho_{2,0})\big) + \big(\alpha-\beta\big)f(\hat{\rho}_2) \\ &\leqslant (1-\alpha)\big(f(\rho_{1,0}) + f(\rho_{2,0})\big) + \big(\alpha-\beta\big)f(\rho_{2,0}) = f(\rho_{4,0}). \end{aligned}$$

Thus (A.1.36) holds. Using (A.1.36) and (A.1.38), we get

$$\begin{aligned} &\mathrm{Tot.Var.}\big(f(\rho(\bar{t}+, \cdot))\big) \\ &= |f(\hat{\rho}_1) - f(\rho_1)| + |f(\hat{\rho}_2) - f(\rho_{2,0})| + |f(\hat{\rho}_3) - f(\rho_{3,0})| + |f(\hat{\rho}_4) - f(\rho_{4,0})| \\ &= \big(f(\hat{\rho}_2) - f(\rho_{2,0})\big) + \big(f(\rho_{3,0}) - f(\hat{\rho}_3)\big) + \big(f(\rho_{4,0}) - f(\hat{\rho}_4)\big) \\ &= f(\rho_{1,0}) - f(\hat{\rho}_1) = f(\rho_{1,0}) - f(\rho_1) = \mathrm{Tot.Var.}\big(f(\rho(\bar{t}-, \cdot))\big). \end{aligned}$$

Suppose now that

$$f(\rho_{1,0}) < f(\rho_1),$$

then $\rho_{1,0} < \rho_1 < \sigma$ and $\Omega_0 \subset \hat{\Omega}$. Assuming again that both points of maximum of the function E are on the curve (A.1.37), we have

$$f(\hat{\rho}_1) = f(\rho_1), \quad f(\hat{\rho}_2) \leqslant f(\rho_{2,0}), \quad f(\rho_{3,0}) = f(\hat{\rho}_3), \quad f(\rho_{4,0}) \leqslant f(\hat{\rho}_4).$$

Therefore we have

$$\begin{aligned}
&\text{Tot.Var.}\big(f(\rho(\bar{t}+,\cdot))\big)\\
&= |f(\hat{\rho}_1) - f(\rho_1)| + |f(\hat{\rho}_2) - f(\rho_{2,0})| + |f(\hat{\rho}_3) - f(\rho_{3,0})| + |f(\hat{\rho}_4) - f(\rho_{4,0})|\\
&= \big(f(\rho_{2,0}) - f(\hat{\rho}_2)\big) + \big(f(\hat{\rho}_3) - f(\rho_{3,0})\big) + \big(f(\hat{\rho}_4) - f(\rho_{4,0})\big)\\
&= f(\hat{\rho}_1) - f(\rho_{1,0}) = f(\rho_1) - f(\rho_{1,0}) = \text{Tot.Var.}\big(f(\rho(\bar{t}-,\cdot))\big).
\end{aligned}$$

This concludes the proof. □

Proof of Lemma 5.3.7 Fix a junction J. Notice that there exists a constant C_J, depending on the coefficients of the matrix A at J, so that each interaction of a wave with J provokes an increase of flux variation at most by a factor C_J. More precisely, if $\text{Tot.Var.}_f^{\pm}$ is the flux variation of waves before and after the interaction then $\text{Tot.Var.}_f^{+} \leqslant C_J \text{Tot.Var.}_f^{-}$.

Consider a wave θ interacting with the junction J. By Lemma A.1.8 the flux variation can increase only if the wave is coming from an outgoing road. Let $\theta_1, \ldots, \theta_4$ be the waves so produced. Thanks to Lemma A.1.6 waves produced by a flux decrease can not interact with the junction J without canceling the flux variation or reaching another junction. Moreover, by Lemma A.1.7, every θ_i can come back to the junction J (without interacting with other junctions) only with a decrease of the flux. By Lemma A.1.4, a wave with decreasing flux interacting with J always produces a flux decrease on outgoing roads. Hence, waves θ_i may come back to the junction only with decreasing flux, thus, by Lemma A.1.6, producing other waves that can not come back to the junction, unless they cancel their flux variation or interact with other junctions. Finally, each wave flux variation can be magnified just twice by a factor C_J interacting only with junction J and not with other junctions.

Now let δ be the minimum length of a road, i.e. $\delta = \min_{i\in\mathcal{I}}(b_i - a_i)$, and $\hat{\lambda}$ be the maximum speed of a wave, i.e. $\hat{\lambda} = \max\{f'(0), |f'(1)|\}$. Then each wave takes at least time $\delta/\hat{\lambda}$ to go from one junction to another.

Finally, recalling that the total variation of the flux may only decrease for interactions on roads, we get that a magnification of flux variation of a factor $C_{\mathcal{J}} = \max_{J\in\mathcal{J}} C_J^2$ may occur only once on each time interval of length $\delta/\hat{\lambda}$. We thus get:

$$\begin{aligned}
\text{Tot.Var.}\big(f(\rho(t+,\cdot))\big) &\leqslant C_{\mathcal{J}}^{\frac{t\hat{\lambda}}{\delta}} \text{Tot.Var.}\big(f(\rho(0+,\cdot))\big) =\\
&= e^{Kt}\text{Tot.Var.}\big(f(\rho(0+,\cdot))\big),
\end{aligned}$$

where $K = \hat{\lambda}\log(C_{\mathcal{J}})/\delta$. □

A.2 Lipschitz Dependence in a Special Case

We present a result about the Lipschitz continuity with respect to initial data.

Let us consider a road network $(\mathcal{I}, \mathcal{J})$.

Definition A.2.1. *Let us fix an approximate wave front tracking solution ρ. For every junction J and for every incoming road I_i, the function $b_\rho(J,i,\cdot)$ is defined on $[0,T]$ by*

$$b_\rho(J,i,t) = \begin{cases} 0, & \text{if } \rho_i(t,b_i-) \in [\sigma,1], \\ 1, & \text{if } \rho_i(t,b_i-) \in [0,\sigma[. \end{cases}$$

If ρ_ν is a sequence of approximate wave front tracking solutions (briefly AWFTS), then we say that the sequence ρ_ν has the property $(\widetilde{H})$ if:

$\widetilde{H}1$. there exists $M \in \mathbb{N}$ such that the function $b_{\rho_\nu}(J,i,\cdot)$ has at most M discontinuities for every $J \in \mathcal{J}$, for every $i \in \{1,\ldots,N\}$ and for every $\nu \geqslant 0$;

$\widetilde{H}2$. there exists $\delta > 0$ such that

$$|\rho_\nu(t,a_i+) - \sigma| > \delta$$

and

$$|\rho_\nu(t,b_i-) - \sigma| > \delta$$

for every $J \in \mathcal{J}$, for every $i \in \{1,\ldots,N\}$, for every $\nu \geqslant 0$ and for every $t \in [0,T]$.

The following proposition holds.

Proposition A.2.2. *Fixed $T > 0$, we consider a solution ρ defined on $[0,T]$ such that, for every $t \in [0,T]$, $\rho(t,\cdot)$ is a bounded variation function. Given $\eta > 0$, $\delta > 0$ and $M \in \mathbb{N}$, we define*

$$\begin{aligned} \mathcal{D}^\eta_\rho(\delta, M) := \{\bar\rho \in L^1_{loc} : \exists (\rho_\nu)_{\nu\in\mathbb{N}} \ & \textit{sequence of AWFTS satisfying } (\widetilde{H}) \\ & \textit{with parameters } \delta \textit{ and } M, \\ & \rho_\nu(0,\cdot) \to \bar\rho(\cdot) \textit{ in } L^1_{loc}, \\ & Tot.Var.(\rho_\nu(0,\cdot) - \rho(0,\cdot)) < \eta\}. \end{aligned}$$

If there exist $0 < \eta' < \eta$, $\delta > 0$ and $M \in \mathbb{N}$ such that

$$\mathcal{D} := cl\,\{\tilde\rho : Tot.Var.(\rho - \tilde\rho) < \eta'\} \subseteq \mathcal{D}^\eta_\rho(\delta, M),$$

then there exists a Lipschitz continuous semigroup S of solutions defined on $[0,T] \times \mathcal{D}$.

6

Aw-Rascle Model on Networks

This chapter, as the previous one, deals with a road network. On each road we consider the Aw-Rascle model for traffic, while at junctions we propose a Riemann solver satisfying the conservation of cars and the following rules:

(A) there are some prescribed preferences of drivers, that is the traffic from incoming roads is distributed on outgoing roads according to fixed coefficients;
(B) respecting (A), drivers choose so as to maximize the first component of the fluxes.

For this model, rules (A) and (B) are not sufficient to isolate a unique solution to the Riemann problem at junctions in outgoing roads. Hence, we impose some additional rules, which permit to solve in a unique way the Riemann problem. Moreover we analyse stability of solutions to the Riemann problem at junctions, in the sense that small variations in the data produce small variations in the solution. Finally, using the wave-front tracking method, we present a result about existence of solutions to the Cauchy problem in a network.

6.1 Basic Definitions and Assumptions

Fix a road network $(\mathcal{I}, \mathcal{J})$. On each road consider the Aw-Rascle model in conservation form

$$\begin{cases} \rho_t + (y - \rho^{\gamma+1})_x = 0, \\ y_t + \left(\frac{y^2}{\rho} - y\rho^{\gamma}\right)_x = 0, \end{cases} \tag{6.1.1}$$

where ρ denotes the car density and y the generalized momentum. As in Chapter 3, consider the domain

$$\mathcal{D} = \left\{(\rho, y) \in \mathbb{R}^+ \times \mathbb{R}^+ : \rho^{\gamma+1} \leqslant y \leqslant \rho\right\}. \tag{6.1.2}$$

For each road I_i, with $(\rho_i, y_i) : [0, +\infty[\times I_i \to \mathcal{D}$ we denote the density and the generalized momentum of cars in the road I_i. For every $i \in \{1, \dots, N\}$, we look for (ρ_i, y_i) in the road I_i, which is an entropy-admissible weak solution, as defined in Chapter 2.

In the following analysis, some curves in the domain $\mathcal{D}$ play a crucial role:

1. the curves of the first family;
2. the curves of the second family;
3. the curve $y = (\gamma + 1)\rho^{\gamma+1}$.

We call the last one *curve of maxima*, since the first component of the flux restricted to a curve of the first family has the maximum point at the intersection with such a curve. Moreover $y = (\gamma + 1)\rho^{\gamma+1}$ divides the domain $\mathcal{D}$ into two subdomains $\mathcal{D}_1$ and $\mathcal{D}_2$:

$$\mathcal{D}_1 := \left\{(\rho, y) \in \mathcal{D} : \ y \geqslant (\gamma + 1)\rho^{\gamma+1}\right\} \tag{6.1.3}$$

and

$$\mathcal{D}_2 := \left\{(\rho, y) \in \mathcal{D} : \ y \leqslant (\gamma + 1)\rho^{\gamma+1}\right\}. \tag{6.1.4}$$

We use the symbols $\mathring{\mathcal{D}}_1$ and $\mathring{\mathcal{D}}_2$ to denote the sets:

$$\mathring{\mathcal{D}}_1 := \left\{(\rho, y) \in \mathcal{D}_1 : \ y > (\gamma + 1)\rho^{\gamma+1}\right\} \tag{6.1.5}$$

and

$$\mathring{\mathcal{D}}_2 := \left\{(\rho, y) \in \mathcal{D}_2 : \ y < (\gamma + 1)\rho^{\gamma+1}\right\} \tag{6.1.6}$$

which are the interior of $\mathcal{D}_1$ and $\mathcal{D}_2$ in the set $\mathcal{D}$.

6.2 Riemann Problems at Junctions

Fix a junction J with n incoming roads (say $I_1, \dots, I_n$) and m outgoing roads (say $I_{n+1}, \dots, I_{n+m}$) and a distribution matrix A satisfying condition (C); see Chapter 5. Assume that $((\rho_{1,0}, y_{1,0}), \dots, (\rho_{n+m,0}, y_{n+m,0}))$ are the initial data on the roads. Our aim is to describe the image of a Riemann solver at the junction J. Some different cases are possible, depending in particular whether the road is incoming or outgoing. Consider a Riemann solver RS satisfying conditions (A) and (B) and define

$$((\hat{\rho}_1, \hat{y}_1), \dots (\hat{\rho}_{n+m}, \hat{y}_{n+m})) = RS((\rho_{1,0}, y_{1,0}), \dots, (\rho_{n+m,0}, y_{n+m,0})). \tag{6.2.7}$$

In the case of an incoming road I_i ($i \in \{1, \dots, n\}$) the following proposition holds.

Proposition 6.2.1. *Let $(\rho_{i,0}, y_{i,0}) \neq (0, 0)$ be the initial value in an incoming road I_i ($i \in \{1, \dots, n\}$). The admissible states $(\hat{\rho}_i, \hat{y}_i)$ generated by the Riemann problem at the junction must belong to the curve of the first family through $(\rho_{i,0}, y_{i,0})$. More precisely, we have the following cases:*

1. $(\rho_{i,0}, y_{i,0}) \in \mathcal{D}_1$. *In this case, the two states are connected by a shock wave of the first family. There exists a unique point* $(\bar{\rho}, \bar{y}) \in \mathcal{D}_2$ *on the curve of the first family through* $(\rho_{i,0}, y_{i,0})$ *with the properties:*
   a) $y_{i,0} - \rho_{i,0}^{\gamma+1} = \bar{y} - \bar{\rho}^{\gamma+1}$;
   b) $(\hat{\rho}_i, \hat{y}_i)$ *is admissible if and only if* $\hat{\rho}_i > \bar{\rho}$; *see Figure 6.1.*
2. $(\rho_{i,0}, y_{i,0}) \in \mathcal{D}_2$. *In this case all the admissible final states belong to* $\mathcal{D}_2$; *see Figure 6.2.*

If instead $(\rho_{i,0}, y_{i,0}) = (0,0)$ *then the only admissible final state is the same point* $(0,0)$.

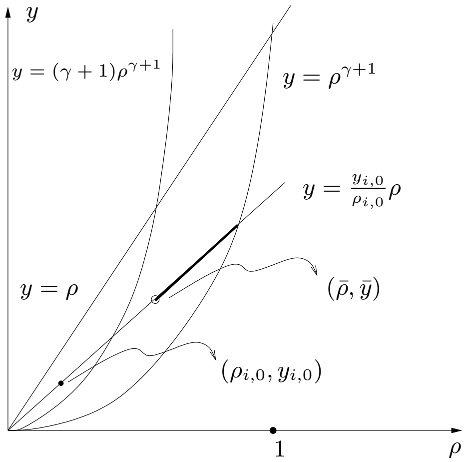


Fig. 6.1. Admissible states in an incoming road I_i when $y_{i,0} > (\gamma+1)\rho_{i,0}^{\gamma+1}$. The final state either is $(\rho_{i,0}, y_{i,0})$ or belongs to the part in bold of the line $y = \frac{y_{i,0}}{\rho_{i,0}}\rho$.

Proof. If we connect two states with a wave of the second family, then the speed of the wave is greater or equal to 0. Therefore, to obtain waves with negative speed one has to restrict to waves of the first family. First, consider the case $(\rho_{i,0}, y_{i,0}) \neq (0,0)$.

If $\hat{\rho}_i < \rho_{i,0}$ then there exists a rarefaction wave of the first family connecting $(\rho_{i,0}, y_{i,0})$ to $(\hat{\rho}_i, \hat{y}_i)$. The maximum speed of the wave is given by

$$\lambda_1(\hat{\rho}_i, \hat{y}_i) = \frac{\hat{y}_i}{\hat{\rho}_i} - (\gamma+1)\hat{\rho}_i^{\gamma}.$$

Since we need $\lambda_1(\hat{\rho}_i, \hat{y}_i) \leqslant 0$, then

$$\hat{\rho}_i^{\gamma+1} \leqslant \hat{y}_i \leqslant (\gamma+1)\hat{\rho}_i^{\gamma+1}.$$

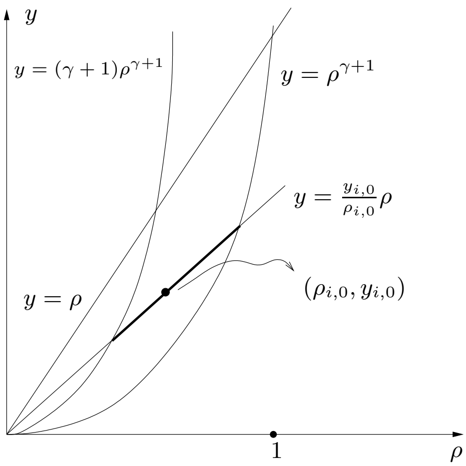


Fig. 6.2. Admissible states in an incoming road I_i when $y_{i,0} < (\gamma+1)\rho_{i,0}^{\gamma+1}$. The final state belongs to the part in bold of the line $y = \frac{y_{i,0}}{\rho_{i,0}}\rho$.

If $\hat{\rho}_i > \rho_{i,0}$ then there exists a shock wave of the first family connecting $(\rho_{i,0}, y_{i,0})$ to $(\hat{\rho}_i, \hat{y}_i)$. Since the speed of the wave, given by the Rankine-Hugoniot condition, must be negative, it results

$$\hat{\rho}_i^{\gamma+1} - \frac{y_{i,0}}{\rho_{i,0}}\hat{\rho}_i + y_{i,0} - \rho_{i,0}^{\gamma+1} > 0.$$

The previous inequality can also be written in the form

$$\frac{y_{i,0}}{\rho_{i,0}} < \frac{\rho_{i,0}^{\gamma+1} - \hat{\rho}_i^{\gamma+1}}{\rho_{i,0} - \hat{\rho}_i}. \tag{6.2.8}$$

If $y_{i,0} \leqslant (\gamma+1)\rho_{i,0}^{\gamma+1}$, then all the points on the curve of the first family through $(\rho_{i,0}, y_{i,0})$ with $\hat{\rho}_i > \rho_{i,0}$ satisfy the last inequality. In fact $y_{i,0}/\rho_{i,0}$ is the slope of the curve of the first family through $(\rho_{i,0}, y_{i,0})$, while

$$\frac{\rho_{i,0}^{\gamma+1} - \hat{\rho}_i^{\gamma+1}}{\rho_{i,0} - \hat{\rho}_i}$$

is strictly greater than the minimum of the derivative of $\rho^{\gamma+1}$ when ρ belongs to the interval

$$\left[\left(\frac{1}{\gamma+1}\right)^{\frac{1}{\gamma}}\left(\frac{y_{i,0}}{\rho_{i,0}}\right)^{\frac{1}{\gamma}}, \left(\frac{y_{i,0}}{\rho_{i,0}}\right)^{\frac{1}{\gamma}}\right],$$

which is exactly $y_{i,0}/\rho_{i,0}$.

Instead, if $y_{i,0} > (\gamma+1)\rho_{i,0}^{\gamma+1}$, then there exists a unique point $(\bar{\rho}, \bar{y})$ on the curve of the first family through $(\rho_{i,0}, y_{i,0})$ with $\bar{\rho} > \rho_{i,0}$ such that

$$\frac{y_{i,0}}{\rho_{i,0}} = \frac{\rho_{i,0}^{\gamma+1} - \bar{\rho}^{\gamma+1}}{\rho_{i,0} - \bar{\rho}}.$$

In fact, since the function $\rho \mapsto \rho^{\gamma+1}$ is convex, then the function

$$\rho \mapsto \frac{\rho_{i,0}^{\gamma+1} - \rho^{\gamma+1}}{\rho_{i,0} - \rho}$$

is strictly increasing when $\rho \geqslant \rho_{i,0}$; moreover

$$\lim_{\rho \to (\frac{1}{\gamma+1})^{1/\gamma}(\frac{y_{i,0}}{\rho_{i,0}})^{1/\gamma}} \frac{\rho_{i,0}^{\gamma+1} - \rho^{\gamma+1}}{\rho_{i,0} - \rho} < \frac{y_{i,0}}{\rho_{i,0}} \quad \text{and} \quad \frac{\rho_{i,0}^{\gamma+1} - (\frac{y_{i,0}}{\rho_{i,0}})^{\frac{\gamma+1}{\gamma}}}{\rho_{i,0} - (\frac{y_{i,0}}{\rho_{i,0}})^{\frac{1}{\gamma}}} > \frac{y_{i,0}}{\rho_{i,0}},$$

gives the existence of $(\bar{\rho}, \bar{y}) \in \mathring{\mathcal{D}}_2$. Notice that the points $(\rho_{i,0}, y_{i,0})$ and $(\bar{\rho}, \frac{y_{i,0}}{\rho_{i,0}}\bar{\rho})$ have the same first component of the flux.

Now, it remains the case $(\rho_{i,0}, y_{i,0}) = (0, 0)$. In this case no point $(\hat{\rho}_i, \hat{y}_i)$ is admissible, since the speed of the wave of the first family connecting $(0, 0)$ to $(\hat{\rho}_i, \hat{y}_i)$ is given by

$$\frac{\hat{y}_i - \hat{\rho}_i^{\gamma+1}}{\hat{\rho}_i},$$

which is clearly positive. Therefore the proof is finished. □

In the case of an outgoing road I_j ($j \in \{n+1, \ldots, n+m\}$) the situation is more complex than the previous one. We describe the images of the Riemann solver RS using an intermediate point $(\bar{\rho}, \bar{y})$.

Proposition 6.2.2. *Any point* $(\bar{\rho}, \bar{y})$ *on a curve of the second family through the point* $(\rho_{j,0}, y_{j,0})$ *can be connected to* $(\rho_{j,0}, y_{j,0})$ *by a contact discontinuity wave of the second family with speed greater than or equal to* 0.

Proof. The proof follows from the fact that the second eigenvalue λ_2 is greater than or equal to 0 in $\mathcal{D}$. □

Proposition 6.2.3. *A state* $(\hat{\rho}_j, \hat{y}_j) \neq (0, 0)$ *is connectible to a given state* $(\bar{\rho}, \bar{y})$ *by a wave of the first family with strictly positive speed if and only if* $\bar{y} = \frac{\hat{y}_j}{\hat{\rho}_j}\bar{\rho}$ *and one of the followings holds:*

1. $\bar{y} < (\gamma+1)\bar{\rho}^{\gamma+1}$. *In this case there exists* $\tilde{\rho} < \bar{\rho}$ *such that all the possible final states* $(\hat{\rho}_j, \hat{y}_j)$ *are those with* $\hat{\rho}_j < \tilde{\rho}$.
2. $\bar{y} \geqslant (\gamma+1)\bar{\rho}^{\gamma+1}$. *In this case we have that*

$$0 \leqslant \hat{\rho}_j \leqslant \left(\frac{1}{\gamma+1}\right)^{1/\gamma} \left(\frac{\hat{y}_j}{\hat{\rho}_j}\right)^{1/\gamma}.$$

If $\hat{\rho}_j < \bar{\rho}$, *then the wave of the first family connecting* $(\hat{\rho}_j, \hat{y}_j)$ *to* $(\bar{\rho}, \bar{y})$ *is a shock wave, while, if* $\hat{\rho}_j > \bar{\rho}$, *then the wave of the first family connecting* $(\hat{\rho}_j, \hat{y}_j)$ *to* $(\bar{\rho}, \bar{y})$ *is a rarefaction wave.*

Proof. First, we note that, if $(\hat{\rho}_j, \hat{y}_j)$ is connectible to $(\bar{\rho}, \bar{y})$ with a wave of the first family, then $\bar{y} = \frac{\hat{y}_j}{\hat{\rho}_j}\bar{\rho}$.

If $\bar{\rho} < \hat{\rho}_j$, then the minimum speed of the wave of the first family connecting $(\hat{\rho}_j, \hat{y}_j)$ to $(\bar{\rho}, \bar{y})$ is given by

$$\lambda_1(\hat{\rho}_j, \hat{y}_j) = \frac{\hat{y}_j}{\hat{\rho}_j} - (\gamma + 1)\hat{\rho}_j^\gamma.$$

Therefore the speed is positive if and only if

$$\hat{y}_j \geqslant (\gamma + 1)\hat{\rho}_j^{\gamma+1}.$$

Instead, if $\bar{\rho} > \bar{\rho}_j$, then the speed of the wave of the first family connecting $(\hat{\rho}_j, \hat{y}_j)$ to $(\bar{\rho}, \bar{y})$ is positive if and only if

$$\frac{(\hat{y}_j - \hat{\rho}_j^{\gamma+1}) - (\bar{y} - \bar{\rho}^{\gamma+1})}{\hat{\rho}_j - \bar{\rho}} > 0,$$

which is equivalent to

$$\frac{\bar{y}}{\bar{\rho}} > \frac{\bar{\rho}^{\gamma+1} - \hat{\rho}_j^{\gamma+1}}{\bar{\rho} - \hat{\rho}_j}. \tag{6.2.9}$$

If $\bar{y} \geqslant (\gamma + 1)\bar{\rho}^{\gamma+1}$, then the supremum of the second member of (6.2.9) when $0 < \hat{\rho}_j < \bar{\rho}$ is equal to $(\gamma + 1)\bar{\rho}^\gamma$, which is lower than or equal to $\bar{y}/\bar{\rho}$.

If instead $\bar{y} < (\gamma+1)\bar{\rho}^{\gamma+1}$, then, as in the proof of Proposition 6.2.1, there exists $\tilde{\rho} < \bar{\rho}$ with the desired property. □

According to Proposition 6.2.1, the first component of the flux for the solution $(\hat{\rho}_i, \hat{y}_i)$ in an incoming road I_i, may take value in the set

$$\Omega_i = \begin{cases} \left[0, \gamma\left(\frac{1}{\gamma+1}\right)^{\frac{\gamma+1}{\gamma}}\left(\frac{y_{i,0}}{\rho_{i,0}}\right)^{\frac{\gamma+1}{\gamma}}\right], & \text{if } (\rho_{i,0}, y_{i,0}) \in \mathcal{D}_2, \\ \left[0, y_{i,0} - \rho_{i,0}^{\gamma+1}\right], & \text{if } (\rho_{i,0}, y_{i,0}) \in \mathcal{D}_1. \end{cases} \tag{6.2.10}$$

By Propositions 6.2.2 and 6.2.3, the first component of the flux for the solution $(\hat{\rho}_j, \hat{y}_j)$ in an outgoing road I_j, may take value in the set

$$\Omega_j = \left[0, \gamma\left(\frac{1}{\gamma+1}\right)^{\frac{\gamma+1}{\gamma}}\right] \tag{6.2.11}$$

if the curve of the second family through $(\rho_{j,0}, y_{j,0})$ is completely inside $\mathcal{D}_1$, while in the set

$$\Omega_j = \left[0, \frac{1}{\rho_{j,0}}(y_{j,0} - \rho_{j,0}^{\gamma+1})\left(1 + \rho_{j,0}^\gamma - \frac{y_{j,0}}{\rho_{j,0}}\right)^{\frac{1}{\gamma}}\right] \tag{6.2.12}$$

in the other case. This situation is also described in Figures 6.3 and 6.4.

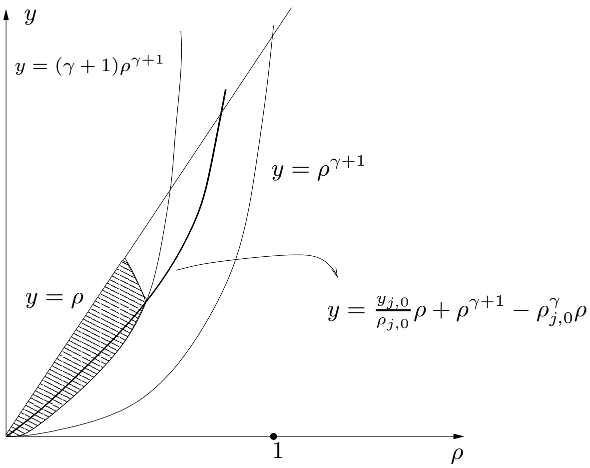


Fig. 6.3. Admissible states in an outgoing road I_j when the curve of the second family through $(\rho_{j,0}, y_{j,0})$ (in bold) is not completely in $\mathcal{D}_1$. The admissible final states $(\hat{\rho}_j, \hat{y}_j)$ belongs to that curve or to the drawn region.

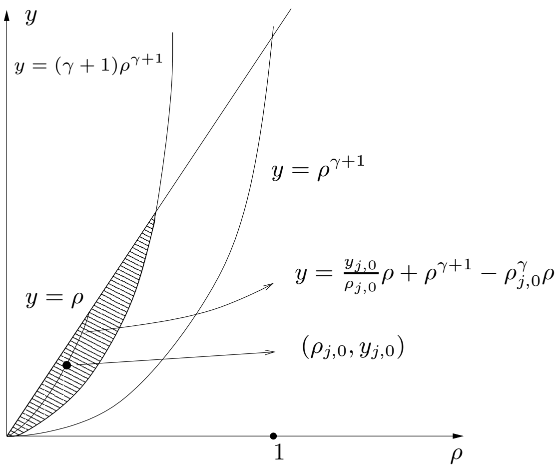


Fig. 6.4. Admissible states in an outgoing road I_j when the curve of the second family through $(\rho_{j,0}, y_{j,0})$ is completely in $\mathcal{D}_1$. The admissible final states $(\hat{\rho}_j, \hat{y}_j)$ belongs to the drawn region.

Remark 6.2.4. Notice that if $(\rho_{j,0}, y_{j,0}) \neq (0,0)$ satisfies $y_{j,0} = \rho_{j,0}^{\gamma+1}$, then the final state $(\hat{\rho}_j, \hat{y}_j)$ must be equal to $(\rho_{j,0}, y_{j,0})$. In fact, if $(\hat{\rho}_j, \hat{y}_j)$ belongs to the curve of the second family through $(\rho_{j,0}, y_{j,0})$, then the wave connecting the two states has zero speed and so it is not admissible, while if $(\hat{\rho}_j, \hat{y}_j)$ belongs to the curve of the first family through $(\rho_{j,0}, y_{j,0})$, then the speed of the wave is negative.

The next theorem gives necessary condition for the Riemann solver RS.

Theorem 6.2.5. *Fix a distributional matrix A satisfying condition (C) and an initial condition $((\rho_{1,0}, y_{1,0}), \dots, (\rho_{n+m,0}, y_{n+m,0}))$. Then there exists a unique $(n+m)$-tuple $(\hat{\delta}_1, \dots, \hat{\delta}_{n+m})$ such that*

1. *$\hat{\delta}_i \geqslant 0$ for every $i \in \{1, \dots, n+m\}$;*
2. *$\hat{y}_i - \hat{\delta}_i^{\gamma+1} = \hat{\delta}_i$ for every $i \in \{1, \dots, n+m\}$.*

Proof. Consider the set

$$\Omega := \{(\delta_1, \dots, \delta_n) \in \Omega_1 \times \dots \times \Omega_n | A \cdot (\delta_1, \dots, \delta_n) \in \Omega_{n+1} \times \dots \times \Omega_{n+m}\}, \tag{6.2.13}$$

where the sets Ω_i are defined in (6.2.10), (6.2.11) and (6.2.12). The set Ω is closed, convex and non empty.

Define the function

$$\begin{array}{rccc} E: & \Omega & \rightarrow & \mathbb{R} \\ & (\delta_1, \dots, \delta_n) & \mapsto & \sum_{i=1}^{n} \delta_i, \end{array} \tag{6.2.14}$$

By condition (C) on the matrix A, ∇E is not orthogonal to any nontrivial subspace contained in a supporting hyperplane to Ω, hence there exists a unique vector $\hat{\delta} = (\hat{\delta}_1, \dots, \hat{\delta}_n) \in \Omega$ such that

$$E(\hat{\delta}_1, \dots, \hat{\delta}_n) = \max_{(\delta_1, \dots, \delta_n) \in \Omega} E(\delta_1, \dots, \delta_n). \tag{6.2.15}$$

With this procedure we find uniquely the values of the fluxes of density of the solution to the Riemann problem at the junction J. More precisely, $\hat{\delta}_i$ gives the value of density fluxes in incoming roads, while density fluxes in outgoing roads are defined by

$$(\hat{\delta}_{n+1}, \dots, \hat{\delta}_{n+m})^T = A \cdot (\hat{\delta}_1, \dots, \hat{\delta}_n)^T.$$

Thus $(\hat{\delta}_1, \dots, \hat{\delta}_{n+m})$ is the unique $(n+m)$-tuple satisfying 1. and 2. in the statement of the theorem. □

Remark 6.2.6. This solution to the Riemann problem at the junction J implies the conservation of the density of car, but does not imply the conservation of the momentum. This means that this kind of solution is not a weak solution to (6.1.1) at J, that is it is not a solution to (6.1.1) in integral sense.

Theorem 6.2.7. *Fix a distributional matrix A satisfying condition (C) and an initial condition* $((\rho_{1,0}, y_{1,0}), \ldots, (\rho_{n+m,0}, y_{n+m,0}))$. *Then, for every* $i \in \{1, \ldots, n\}$, *there exists a unique couple* $(\hat{\rho}_i, \hat{y}_i)$ *satisfying (6.2.7).*

Proof. Fix $i \in \{1, \ldots, n\}$. By Theorem 6.2.5 we have to choose an element $(\hat{\rho}_i, \hat{y}_i)$, satisfying the necessary conditions given by Proposition 6.2.1 and

$$\hat{y}_i - \hat{\rho}_i^{\gamma+1} = \hat{\delta}_i.$$

Therefore, we need to solve the system

$$\begin{cases} y = \frac{y_{i,0}}{\rho_{i,0}} \rho, \\ y = \rho^{\gamma+1} + \hat{\delta}_i. \end{cases} \tag{6.2.16}$$

This system in general admits two solutions in $\mathcal{D}$, but only one satisfies the conditions of Proposition 6.2.1. So we take

$$(\hat{\rho}_i, \hat{y}_i) = (\rho_{i,0}, y_{i,0}) \tag{6.2.17}$$

if $y_{i,0} = \rho_{i,0}^{\gamma+1} + \hat{\delta}_i$, otherwise $(\hat{\rho}_i, \hat{y}_i)$ is the unique solution in $\mathcal{D}_2$ of the system (6.2.16). □

Remark 6.2.8. Condition on the speed of waves in outgoing roads implies, as proved in Propositions 6.2.2 and 6.2.3, that $(\hat{\rho}_j, \hat{y}_j)$ may stay in a two-dimensional region; see also Figures 6.3 and 6.4. Theorem 6.2.5 implies also that

$$\hat{y}_j - \hat{\rho}_j^{\gamma+1} = \hat{\delta}_j.$$

In general, these conditions are not sufficient to isolate a unique $(\hat{\rho}_j, \hat{y}_j)$. To do this, we need to impose some additional conditions.

Consider three different additional rules for solutions to a Riemann problem at the junction J:

(AR-1) maximize the velocity v of cars in outgoing roads;
(AR-2) maximize the density ρ of cars in outgoing roads;
(AR-3) minimize the total variation of ρ along the solution of the Riemann problem in outgoing roads.

Remark 6.2.9. Rules (AR-1) and (AR-2) are given for model reason, assuming that drivers prefer to maximize ρ or v. On the other side rule (AR-3) is motivated mathematically to control the BV norm.

6.2.1 (AR-1): Maximize the Speed

This subsection deals with the solution to a Riemann problem at the junction J with the additional rule (AR-1). With this rule it is possible to isolate a unique solution also for the outgoing roads. Indeed the next theorem holds.

Theorem 6.2.10. *Fix a distributional matrix A satisfying condition (C) and an initial condition $((\rho_{1,0}, y_{1,0}), \dots, (\rho_{n+m,0}, y_{n+m,0}))$. Then, for every $j \in \{n+1, \dots, n+m\}$, there exists a unique couple $(\hat{\rho}_j, \hat{y}_j)$ satisfying (6.2.7) and the additional rule (AR-1). Moreover, the point $(\hat{\rho}_j, \hat{y}_j)$ belongs to the line $y = \rho$.*

Proof. First, recall that the second characteristic field is linearly degenerate and the second eigenvalue λ_2 is equal to the velocity v of the cars. So the curves of the second family are the level sets for the speed v of the cars. Moreover, the speed v is monotone decreasing with respect to ρ on level curves of density flux. Therefore, $(\hat{\rho}_j, \hat{y}_j)$ is the solution to

$$\begin{cases} y = \rho^{\gamma+1} + \hat{\delta}_j, \\ y = \rho. \end{cases} \tag{6.2.18}$$

There are some different cases.

1. $\hat{\delta}_j < \sup \Omega_j$. In this case $(\hat{\rho}_j, \hat{y}_j)$ is the solution to the system (6.2.18), that is in $\mathcal{D}_1$. In general, to connect $(\hat{\rho}_j, \hat{y}_j)$ with $(\rho_{j,0}, y_{j,0})$ we use a wave of the first family with positive speed and a wave of the second family; see Figure 6.5.

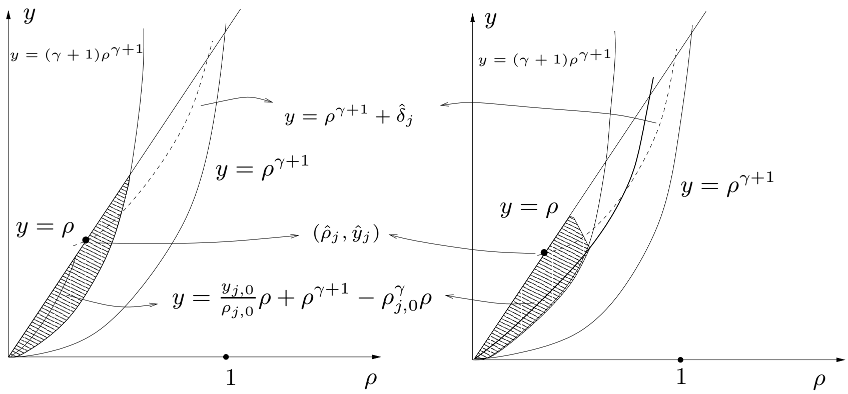


Fig. 6.5. Solution $(\hat{\rho}_j, \hat{y}_j)$ to the Riemann problem on an outgoing road I_j in the case 1 with the additional rule (AR-1). In the first picture it is drawn the case in which the curve $y = \rho^{\gamma+1} + \hat{\delta}_j$ does not intersect in $\mathcal{D}$ the curve of the second family through $(\rho_{j,0}, y_{j,0})$, while in the second picture the other case.

2. $\hat{\delta}_j = \sup(\Omega_j)$, $y_{j,0} < \rho_{j,0}$, and the curve of the second family through $(\rho_{j,0}, y_{j,0})$ lies completely in the region $\mathcal{D}_1$; see Figure 6.6. In this case the set Ω_j is given by (6.2.11) and so it is the maximum possible. Hence there exists only one point in $\mathcal{D}$ with the first component of the flux equal to $\hat{\delta}_j$ and this point is precisely given by the intersection between the line $y = \rho$ and the curve of maxima. Thus

$$(\hat{\rho}_j, \hat{y}_j) = \left(\left(\frac{1}{\gamma+1}\right)^{\frac{1}{\gamma}}, \left(\frac{1}{\gamma+1}\right)^{\frac{1}{\gamma}} \right),$$

and to connect $(\hat{\rho}_j, \hat{y}_j)$ with $(\rho_{j,0}, y_{j,0})$ we may use a wave of the first family with positive speed and a wave of the second family.

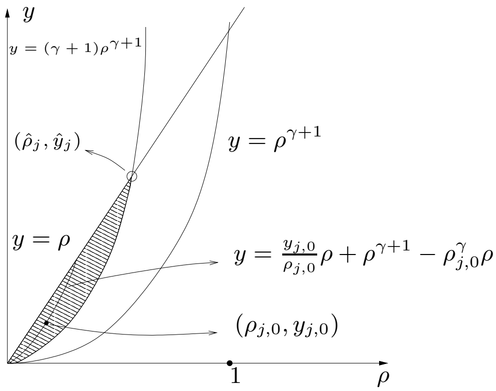


Fig. 6.6. Solution $(\hat{\rho}_j, \hat{y}_j)$ to the Riemann problem on an outgoing road I_j in the case 2 with the additional rule (AR-1).

3. $\hat{\delta}_j = \sup(\Omega_j)$, $y_{j,0} < \rho_{j,0}$ and the curve of the second family through $(\rho_{j,0}, y_{j,0})$ is not completely contained in the region $\mathcal{D}_1$. In this case $(\hat{\rho}_j, \hat{y}_j)$ is given by the solution to

$$\begin{cases} y = \rho, \\ y = \rho^{\gamma+1} + \hat{\delta}_j, \\ (\rho, y) \in \mathcal{D}_2, \end{cases}$$

since as in the previous case the intersection between the region of admissible final states and the curve $y = \rho^{\gamma+1} + \hat{\delta}_j$ consists of a single point; see Figure 6.7. To connect $(\hat{\rho}_j, \hat{y}_j)$ with $(\rho_{j,0}, y_{j,0})$ we use only a wave of the second family.

4. $\hat{\delta}_j = \sup(\Omega_j)$ and $y_{j,0} = \rho_{j,0}$. If $(\rho_{j,0}, y_{j,0}) \in \mathcal{D}_1$, then as in case 2 the set Ω_j is the maximum possible and so the solution is given by

$$(\hat{\rho}_j, \hat{y}_j) = \left(\left(\frac{1}{\gamma+1}\right)^{\frac{1}{\gamma}}, \left(\frac{1}{\gamma+1}\right)^{\frac{1}{\gamma}} \right),$$

and to connect $(\hat{\rho}_j, \hat{y}_j)$ with $(\rho_{j,0}, y_{j,0})$ we use only a wave of the first family with positive speed. If instead $(\rho_{j,0}, y_{j,0}) \in \mathcal{D}_2$, then, as in case 3, the solution

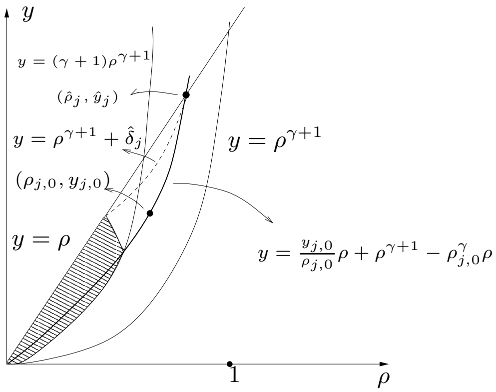


Fig. 6.7. Solution $(\hat{\rho}_j, \hat{y}_j)$ to the Riemann problem on an outgoing road I_j in the case 3 with the additional rule (AR-1).

is

$$(\hat{\rho}_j, \hat{y}_j) = (\rho_{j,0}, y_{j,0}),$$

and no wave is produced. This case completes the proof. □

Remark 6.2.11. Notice that, in the case 1. of the previous proof, if we recalculate Ω_j using $(\hat{\rho}_j, \hat{y}_j)$ instead of $(\rho_{j,0}, y_{j,0})$, then the obtained set $\hat{\Omega}_j$ may be bigger than Ω_j. Thus it seems that the solution can be found in two steps, but this is not the case, since $\hat{\delta}_j < \sup(\Omega_j)$ implies that the maximization problem (6.2.15) has the same solution.

Remark 6.2.12. The solution to the Riemann problem at J with the additional rule (AR-1) is equal to the solution to the Riemann problem at J with the rule: minimize the density of the cars in outgoing roads.

6.2.2 (AR-2): Maximize the Density

This subsection deals with the solution to a Riemann problem at the junction J with the additional rule (AR-2). The following theorem holds.

Theorem 6.2.13. *Fix a distributional matrix A satisfying condition (C) and an initial condition $((\rho_{1,0}, y_{1,0}), \ldots, (\rho_{n+m,0}, y_{n+m,0}))$. Then, for every $j \in \{n+1, \ldots, n+m\}$, there exists a unique couple $(\hat{\rho}_j, \hat{y}_j)$ satisfying (6.2.7) and the additional rule (AR-2). Moreover, the point $(\hat{\rho}_j, \hat{y}_j)$ belongs to the region $\mathcal{D}_2$ and either to the curve of the second family through $(\rho_{j,0}, y_{j,0})$ or to the curve of maxima.*

Proof. We have two different possibilities.

1. The curve of the second family through $(\rho_{j,0}, y_{j,0})$ is completely in the region $\mathcal{D}_1$. In this case the admissible final states are exactly all the points of

$\mathcal{D}_1$ and so the solution $(\hat{\rho}_j, \hat{y}_j)$ belongs to the part of the curve $y = \rho^{\gamma+1} + \hat{\delta}_j$ which lies in $\mathcal{D}_1$. If we want to maximize the density we have to choose the point $(\hat{\rho}_j, \hat{y}_j)$ given by

$$\begin{cases} y = \rho^{\gamma+1} + \hat{\delta}_j, \\ y = (\gamma+1)\rho^{\gamma+1}, \end{cases}$$

as we clearly see in figure 6.8.a. To connect $(\hat{\rho}_j, \hat{y}_j)$ with $(\rho_{j,0}, y_{j,0})$, we have to use in general a wave of the first family with positive speed and a wave of the second family.

2. The curve of the second family through $(\rho_{j,0}, y_{j,0})$ is not completely in the region $\mathcal{D}_1$. If the curve of the second family through $(\rho_{j,0}, y_{j,0})$ intersects the curve $y = \rho^{\gamma+1} + \hat{\delta}_j$ only in the region $\mathcal{D}_1$, then the solution $(\hat{\rho}_j, \hat{y}_j)$ is given as in the previous case by the system

$$\begin{cases} y = \rho^{\gamma+1} + \hat{\delta}_j, \\ y = (\gamma+1)\rho^{\gamma+1}, \end{cases}$$

and to connect the two states we use a wave of the first family with positive speed and a wave of the second family. Otherwise $(\hat{\rho}_j, \hat{y}_j)$ is given solving

$$\begin{cases} y = \rho^{\gamma+1} + \hat{\delta}_j, \\ y = \frac{y_{j,0}}{\rho_{j,0}}\rho + \rho^{\gamma+1} - \rho_{j,0}^{\gamma}\rho, \end{cases}$$

as we see in figure 6.8.b. To connect $(\hat{\rho}_j, \hat{y}_j)$ with $(\rho_{j,0}, y_{j,0})$, we use only a wave of the second family. □

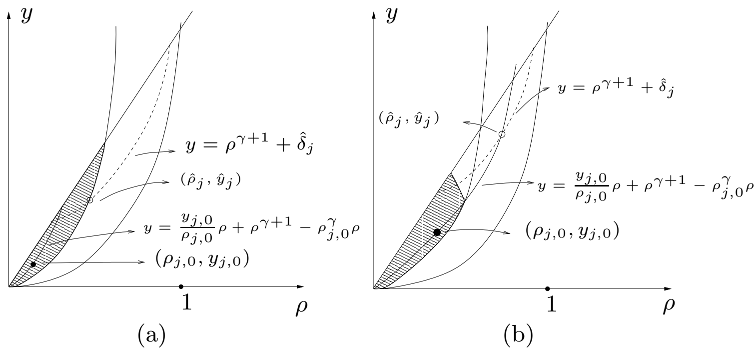


Fig. 6.8. Solution $(\hat{\rho}_j, \hat{y}_j)$ to the Riemann problem on an outgoing road I_j with the additional rule (AR-3). The first figure shows the case where the curve of the second family through $(\rho_{j,0}, y_{j,0})$ is completely in the region $\mathcal{D}_1$, while the second figure shows the other case.

6.2.3 (AR-3): Minimize the Total Variation

This subsection deals with the solution to a Riemann problem at the junction J with the additional rule (AR-3). The following theorem holds.

Theorem 6.2.14. *Fix a distributional matrix A satisfying condition (C) and an initial condition $((\rho_{1,0}, y_{1,0}), \dots, (\rho_{n+m,0}, y_{n+m,0}))$. Then, for every $j \in \{n+1, \dots, n+m\}$, there exists a unique couple $(\hat{\rho}_j, \hat{y}_j)$ satisfying (6.2.7) and the additional rule (AR-3).*

This theorem follows directly by the following lemmata.

Lemma 6.2.15. *If $(\rho_{j,0}, y_{j,0}) = (0, 0)$, then the point $(\hat{\rho}_j, \hat{y}_j)$ belongs to the line $y = \rho$ and to the region $\mathcal{D}_1$.*

Proof. Here the set of admissible states is the whole $\mathcal{D}$ and in this case minimizing the total variation of ρ along a solution is equivalent to choose the point of the curve $y = \rho^{\gamma+1} + \hat{\delta}_j$ with minimum ρ. □

From Remark 6.2.4, we deduce immediately the following lemma.

Lemma 6.2.16. *Let $(\rho_{j,0}, y_{j,0}) \neq (0, 0)$ and $y_{j,0} = \rho_{j,0}^{\gamma+1}$. In this case $(\hat{\rho}_j, \hat{y}_j)$ is equal to $(\rho_{j,0}, y_{j,0})$.*

Lemma 6.2.17. *Assume $y_{j,0} > \rho_{j,0}^{\gamma+1}$. If the curve $y = \rho^{\gamma+1} + \hat{\delta}_j$ intersects the curve of the second family through $(\rho_{j,0}, y_{j,0})$, then $(\hat{\rho}_j, \hat{y}_j)$ is given by the unique intersection of those curves.*

Proof. It is easy to see that the intersection between the curve of the second family through $(\rho_{j,0}, y_{j,0})$ and the curve $y = \rho^{\gamma+1} + \hat{\rho}_j$ consists of at most one point. Then $(\hat{\rho}_j, \hat{y}_j)$ is such intersection and the total variation of the density ρ along the solution is simply given by $|\hat{\rho}_j - \rho_{j,0}|$. In order to prove that the solution attains the minimum of variation in ρ, let us consider an other admissible point $(\bar{\rho}, \bar{y})$ such that

$$\begin{cases} \bar{y} = \bar{\rho}^{\gamma+1} + \hat{\delta}_j, \\ \bar{\rho} \neq \hat{\rho}_j. \end{cases}$$

We must have $\min\{\hat{\rho}_j, \rho_{j,0}\} \leqslant \bar{\rho} \leqslant \max\{\hat{\rho}_j, \rho_{j,0}\}$. If such a point $(\bar{\rho}, \bar{y})$ exists, then to connect $(\bar{\rho}, \bar{y})$ with $(\rho_{j,0}, y_{j,0})$ we need to use first a wave of the first family until a point $(\tilde{\rho}, \tilde{y})$ and then a wave of the second family. Thus the total variation of the density ρ along this solution is given by $|\bar{\rho} - \tilde{\rho}| + |\tilde{\rho} - \rho_{j,0}|$, where $\tilde{\rho}$ satisfies either $\tilde{\rho} < \min\{\hat{\rho}_j, \rho_{j,0}\}$ or $\tilde{\rho} > \max\{\hat{\rho}_j, \rho_{j,0}\}$ and so the proof is finished. □

Lemma 6.2.18. *Assume $y_{j,0} > \rho_{j,0}^{\gamma+1}$. If the curve $y = \rho^{\gamma+1} + \hat{\delta}_j$ does not intersect the curve of the second family through $(\rho_{j,0}, y_{j,0})$, then $(\hat{\rho}_j, \hat{y}_j)$ is given by*

$$\begin{cases} y = \rho^{\gamma+1} + \hat{\delta}_j, \\ y = \rho, \\ (\rho, y) \in \mathcal{D}_1. \end{cases} \tag{6.2.19}$$

Proof. The only possibility is that the curve of the second family through $(\rho_{j,0}, y_{j,0})$ is completely inside the region $\mathcal{D}_1$. Let us call $(\hat{\rho}_j, \hat{y}_j)$ the solution to (6.2.19). To connect $(\hat{\rho}_j, \hat{y}_j)$ with $(\rho_{j,0}, y_{j,0})$ we have to use first a rarefaction wave of the first family until a state $(\bar{\rho}, \bar{y})$ with $\rho_{j,0} < \bar{\rho} < \hat{\rho}_j$ and then a wave of the second family. The total variation of the density ρ along this solution is equal to $|\hat{\rho}_j - \rho_{j,0}|$. Any other point of $y = \rho^{\gamma+1} + \hat{\delta}_j$ generates a variation in ρ strictly bigger than $|\hat{\rho}_j - \rho_{j,0}|$ and so the lemma is proved. □

6.3 Stability of Solutions to Riemann Problems at Junctions

The aim of this section is to investigate stability of constant (on each road) solutions to Riemann problem, called equilibria. Stability simply means that small perturbations of the data in L^∞ norm, that may be produced by waves arriving at junctions, produce small variations of the equilibrium in L^∞ norm. As in the previous section, we have to consider different cases according to the additional rules (AR-1), (AR-2) or (AR-3).

In the whole section, we consider a fixed junction J with n incoming roads (say $I_1, \ldots, I_n$) and m outgoing roads (say $I_{n+1}, \ldots, I_{n+m}$) and we assume that $((\rho_{1,0}, y_{1,0}), \ldots, (\rho_{n+m,0}, y_{n+m,0}))$ is an equilibrium at J.

Definition 6.3.1. *We say that* $((\rho_{1,0}, y_{1,0}), \ldots, (\rho_{n+m,0}, y_{n+m,0}))$ *is an equilibrium at J if*

$$((\rho_{1,0}, y_{1,0}), \ldots, (\rho_{n+m,0}, y_{n+m,0})) = RS((\rho_{1,0}, y_{1,0}), \ldots, (\rho_{n+m,0}, y_{n+m,0})).$$

We want to remark that waves of the second family have always positive speed. Moreover waves of the first family connecting two states in the region $\mathcal{D}_1$ have positive speed, while waves of the first family connecting two states in the region $\mathcal{D}_2$ have negative speed. The consequences of this fact are the followings.

Claim 1. In an outgoing road only waves of the first family can reach the junction. Therefore if $(\rho_{j,0}, y_{j,0}) \in \mathring{\mathcal{D}}_1$, then it can not be perturbed by waves connecting $(\rho_{j,0}, y_{j,0})$ with an other state $(\bar{\rho}, \bar{y}) \in \mathring{\mathcal{D}}_1$; in fact, in this case, also waves of the first family have positive speed.

Claim 2. Assume $(\rho_{i,0}, y_{i,0}) \in \mathring{\mathcal{D}}_1$. If a wave on a road different from I_i produces a variation of the solution of the Riemann problem at the junction, then the new solution $(\hat{\rho}_i, \hat{y}_i)$ in the incoming road I_i either is equal to $(\rho_{i,0}, y_{i,0})$ or $(\hat{\rho}_i, \hat{y}_i)$ belongs to $\mathring{\mathcal{D}}_2$. In the latter case the distance between $(\rho_{i,0}, y_{i,0})$ and $(\hat{\rho}_i, \hat{y}_i)$ is proportional to the distance between $(\rho_{i,0}, y_{i,0})$ and the curve of maxima. Thus, such configuration is unstable.

6.3.1 (AR-1): Maximize the Speed

Recall that, by Theorem 6.2.10, all equilibria for outgoing roads must belong to the line $y = \rho$. The analysis of all equilibria is very complicated, hence we prefer to treat in detail only some significant cases. We also consider all the general case when $n = m = 2$.

We have some different possibilities.

1. $(\rho_{j,0}, y_{j,0}) \in \mathring{\mathcal{D}}_1$ for every $j \in \{n+1, \ldots, n+m\}$. Therefore, the maximization problem (6.2.15) implies that $(\rho_{i,0}, y_{i,0}) \in \mathcal{D}_1$ for every $i \in \{1, \ldots, n\}$. In this case, by (6.2.10) and (6.2.11), we deduce that: for incoming roads

$$\Omega_i = \left[0, y_{i,0} - \rho_{i,0}^{\gamma+1}\right],$$

while for outgoing ones

$$\Omega_j = \left[0, \gamma \left(\frac{1}{\gamma+1}\right)^{\frac{\gamma+1}{\gamma}}\right].$$

If we denote by $\delta_{i,0} := y_{i,0} - \rho_{i,0}^{\gamma+1}$ for every $i \in \{1, \ldots, n\}$ and by $\delta_{j,0} := y_{j,0} - \rho_{j,0}^{\gamma+1}$ for every $j \in \{n+1, \ldots, n+m\}$, then clearly $(\delta_{1,0}, \ldots, \delta_{n,0})$ is the solution of the maximization problem (6.2.15) and

$$(\delta_{n+1,0}, \ldots, \delta_{n+m,0})^T = A \cdot (\delta_{1,0}, \ldots, \delta_{n,0})^T.$$

The hypothesis $y_{j,0} > (\gamma+1)\rho_{j,0}^{\gamma+1}$ for every $j \in \{n+1, \ldots, n+m\}$ has the following two consequences. Firstly, $\delta_{j,0} < \sup \Omega_j$ and hence the outgoing roads give no constraint for the maximization problem (6.2.15). Secondly, by claim 1, the outgoing roads cannot be perturbed by waves with negative speed. Consider a perturbation produced by a wave of the first or second family from an incoming road I_i connecting $(\tilde{\rho}_i, \tilde{y}_i)$ with $(\rho_{i,0}, y_{i,0})$. The possible density fluxes are in the set

$$\widetilde{\Omega}_i = \left[0, \tilde{y}_i - \tilde{\rho}_i^{\gamma+1}\right]$$

if $(\tilde{\rho}_i, \tilde{y}_i) \in \mathcal{D}_1$, while

$$\widetilde{\Omega}_i = \left[0, \gamma \left(\frac{1}{\gamma+1}\right)^{\frac{\gamma+1}{\gamma}} \left(\frac{\tilde{y}_i}{\tilde{\rho}_i}\right)^{\frac{\gamma+1}{\gamma}}\right]$$

in the other case. Since the outgoing roads are not constraints for the maximization problem (6.2.15), we may suppose the following, provided the perturbation is sufficiently small:

(a) the new maximum point for (6.2.15) is

$$(\hat{\delta}_1, \ldots, \hat{\delta}_n) := \left(y_{1,0} - \rho_{1,0}^{\gamma+1}, \ldots, \tilde{y}_i - \tilde{\rho}_i^{\gamma+1}, \ldots, y_{n,0} - \rho_{n,0}^{\gamma+1}\right)$$

if $(\tilde{\rho}_i, \tilde{y}_i) \in \mathcal{D}_1$, while

$$(\hat{\delta}_1, \ldots, \hat{\delta}_n) := \left(y_{1,0} - \rho_{1,0}^{\gamma+1}, \ldots, \gamma \left(\frac{1}{\gamma+1} \right)^{\frac{\gamma+1}{\gamma}} \left(\frac{\tilde{y}_i}{\tilde{\rho}_i} \right)^{\frac{\gamma+1}{\gamma}}, \ldots, y_{n,0} - \rho_{n,0}^{\gamma+1} \right).$$

in the other case;

(b) the solution $(\hat{\delta}_{n+1}, \ldots, \hat{\delta}_{n+m})$ defined by

$$(\hat{\delta}_{n+1}, \ldots, \hat{\delta}_{n+m})^T = A \cdot (\hat{\delta}_1, \ldots, \hat{\delta}_n)^T$$

satisfies

$$\hat{\delta}_j < \sup \widetilde{\Omega}_j$$

for every $j \in \{n+1, \ldots, n+m\}$ (the outgoing roads do not become constraints for the maximization problem (6.2.15));

(c) there exists a positive constant C such that

$$|(\hat{\rho}_i, \hat{y}_i) - (\tilde{\rho}_i, \tilde{y}_i)| + \sum_{j=n+1}^{n+m} |(\hat{\rho}_j, \hat{y}_j) - (\rho_{j,0}, y_{j,0})| < C \, |(\rho_i, y_i) - (\tilde{\rho}_i, \tilde{y}_i)| .$$

Moreover in outgoing roads waves of the first family are produced, while in incoming roads no waves are produced except in the I_i road.

The conclusion is that this kind of equilibrium is stable under small perturbations.

2. $(\rho_{i,0}, y_{i,0}) \in \mathring{\mathcal{D}}_1$ for every $i \in \{1, \ldots, n\}$ and $(\rho_{j,0}, y_{j,0}) \in \mathcal{D}_2$ for some $j \in \{n+1, \ldots, n+m\}$. This is an unstable equilibrium. In fact, let I_{j_1} be the outgoing road with the property $y_{j_1,0} \leqslant (\gamma+1)\rho_{j_1,0}^{\gamma+1}$. It is possible to consider a perturbation generated by a wave of the first family connecting $(\rho_{j_1,0}, y_{j_1,0})$ with $(\tilde{\rho}_{j_1}, \tilde{y}_{j_1})$ such that

$$\sup \widetilde{\Omega}_{j_1} < \sup \Omega_{j_1},$$

where $\widetilde{\Omega}_{j_1}$ is defined as in (6.2.12) for the state $(\tilde{\rho}_{j_1}, \tilde{y}_{j_1})$. In this case the maximization problem (6.2.15) produces a flux in an incoming road I_i, which is strictly lower than $\sup \Omega_i$, hence the final state jumps into the region $\mathring{\mathcal{D}}_2$.

3. $(\rho_{i,0}, y_{i,0}) \in \mathring{\mathcal{D}}_2$ for every $i \in \{1, \ldots, n\}$ and $(\rho_{j,0}, y_{j,0}) \in \mathring{\mathcal{D}}_2$ for every $j \in \{n+1, \ldots, n+m\}$. The fact that $y_{i,0} < (\gamma+1)\rho_{i,0}^{\gamma+1}$ for every $i \in \{1, \ldots, n\}$ implies that $\delta_{i,0} := y_{i,0} - \rho_{i,0}^{\gamma+1} < \sup \Omega_i$ for every $i \in \{1, \ldots, n\}$ and hence the incoming roads are not constraints for the maximization problem (6.2.15). Therefore we have stability for perturbations by waves from incoming roads.

Instead the perturbation of an outgoing road in general produces a variation of the maximization problem (6.2.15), since by hypotheses $\delta_{j,0} := y_{j,0} - \rho_{j,0}^{\gamma+1} = \sup \Omega_j$ for every $j \in \{n+1, \ldots, n+m\}$ (all the outgoing roads are constraints for the maximization problem (6.2.15)).

First of all, let us consider the case $m > n$. The maximum for (6.2.15) is determined only by n constraints. Consider a wave of the first family in an outgoing road I_j connecting $(\rho_{j,0}, y_{j,0})$ with $(\tilde{\rho}_j, \tilde{y}_j) \in \mathring{\mathcal{D}}_2$. We denote by $\widetilde{\Omega}_j$ the set defined by (6.2.12) where $(\tilde{\rho}_j, \tilde{y}_j)$ is the initial state. If $\sup \widetilde{\Omega}_j > \sup \Omega_j$, then the maximum for (6.2.15) does not vary, but the I_j road is no more an active constraint since $\delta_j < \sup \widetilde{\Omega}_j$. Then the final state $(\hat{\rho}_j, \hat{y}_j) \in \mathring{\mathcal{D}}_1$ and the equilibrium is unstable.

Now let us consider the case $m = n$. We consider a perturbation in an outgoing road I_j by a wave of the first family connecting $(\rho_{j,0}, y_{j,0})$ with $(\tilde{\rho}_j, \tilde{y}_j)$. If the perturbation is sufficiently small, then we may suppose the following:

(a) the new solution $(\hat{\delta}_1, \dots, \hat{\delta}_n)$ of the maximization problem (6.2.15) satisfies $\hat{\delta}_i < \sup \Omega_i$ for every $i \in \{1, \dots, n\}$ (the incoming roads are not constraints for the maximization problem (6.2.15)) and the final states $(\hat{\rho}_i, \hat{y}_i)$ in the incoming roads belong to $\mathring{\mathcal{D}}_2$;
(b) $\hat{\delta}_j = \tilde{y}_j - \tilde{\rho}_j^{\gamma+1}$, the fluxes for the other outgoing roads remain the same and the final state $(\hat{\rho}_j, \hat{y}_j)$ in the I_j outgoing road coincides with $(\tilde{\rho}_j, \tilde{y}_j)$;
(c) there exists a positive constant C such that

$$\sum_{i=1}^{n} |(\rho_{i,0}, y_{i,0}) - (\hat{\rho}_i, \hat{y}_i)| < C \, |(\rho_{j,0}, y_{j,0}) - (\tilde{\rho}_j, \tilde{y}_j)| \, .$$

Therefore the equilibrium is stable.

We may summarize all these results in the following.

Theorem 6.3.2. *If $(\rho_{j,0}, y_{j,0}) \in \mathring{\mathcal{D}}_1$ for every $j \in \{n+1, \dots, n+m\}$, then the equilibrium is stable.*

If $m = n$, $(\rho_{i,0}, y_{i,0}) \in \mathring{\mathcal{D}}_2$ for every $i \in \{1, \dots, n\}$ and $(\rho_{j,0}, y_{j,0}) \in \mathring{\mathcal{D}}_2$ for every $j \in \{n+1, \dots, 2n\}$, then the equilibrium is stable.

Consider now the generic case for $m = n = 2$. For generic we mean that the active constraints are given exactly by two roads and the states belong to $\mathring{\mathcal{D}}_1$ and $\mathring{\mathcal{D}}_2$.

1. $(\rho_{1,0}, y_{1,0}) \in \mathring{\mathcal{D}}_1$, $(\rho_{2,0}, y_{2,0}) \in \mathring{\mathcal{D}}_1$, $(\rho_{3,0}, y_{3,0}) \in \mathring{\mathcal{D}}_1$, $(\rho_{4,0}, y_{4,0}) \in \mathring{\mathcal{D}}_1$. This case is covered by the previous theorem.

2. $(\rho_{1,0}, y_{1,0}) \in \mathring{\mathcal{D}}_1$, $(\rho_{2,0}, y_{2,0}) \in \mathring{\mathcal{D}}_2$, $(\rho_{3,0}, y_{3,0}) \in \mathring{\mathcal{D}}_2$, $(\rho_{4,0}, y_{4,0}) \in \mathring{\mathcal{D}}_1$. In this case the active constraints are given by the roads I_1 and I_3. By claim 1, we know that the datum $(\rho_{4,0}, y_{4,0})$ can not be perturbed. Consider a perturbation produced by a wave of the second family connecting $(\tilde{\rho}_2, \tilde{y}_2)$ with $(\rho_{2,0}, y_{2,0})$. If the strength of the wave is sufficiently small, then the maximization problem (6.2.15) admits the same maximum point. Therefore no change happens in road I_1 and I_3, and

$$|(\tilde{\rho}_2, \tilde{y}_2) - (\hat{\rho}_2, \hat{y}_2)| + |(\rho_{4,0}, y_{4,0}) - (\hat{\rho}_4, \hat{y}_4)| \leqslant C \, |(\tilde{\rho}_2, \tilde{y}_2) - (\rho_{2,0}, y_{2,0})| \, ,$$

where C is a positive constant and $(\hat{\rho}_2, \hat{y}_2)$ and $(\hat{\rho}_4, \hat{y}_4)$ are the final states respectively in roads I_2 and I_4.

Consider now a perturbation produced by a wave connecting $(\tilde{\rho}_1, \tilde{y}_1)$ with $(\rho_{1,0}, y_{1,0})$. We may suppose the followings, provided the perturbation is small:

(a) the active constraints remain the roads I_1 and I_3;
(b) the final state in I_3 is $(\hat{\rho}_3, \hat{y}_3) = (\rho_{3,0}, y_{3,0})$, while in I_1 is $(\hat{\rho}_1, \hat{y}_1) = (\tilde{\rho}_1, \tilde{y}_1)$;
(c) for some $C > 0$ we have

$$|(\hat{\rho}_2, \hat{y}_2) - (\rho_{2,0}, y_{2,0})| + |(\rho_{4,0}, y_{4,0}) - (\hat{\rho}_4, \hat{y}_4)| \leqslant C\,|(\tilde{\rho}_1, \tilde{y}_1) - (\rho_{1,0}, y_{1,0})|\,,$$

where $(\hat{\rho}_2, \hat{y}_2)$ and $(\hat{\rho}_4, \hat{y}_4)$ are the final states respectively in roads I_2 and I_4.

The case of a perturbation in I_3 is completely similar. Therefore this equilibrium is stable. The other cases, in which the active constraints are given by an incoming road and an outgoing road, are similar to this one and so stable.

3. $(\rho_{1,0}, y_{1,0}) \in \mathring{\mathcal{D}}_2$, $(\rho_{2,0}, y_{2,0}) \in \mathring{\mathcal{D}}_2$, $(\rho_{3,0}, y_{3,0}) \in \mathring{\mathcal{D}}_2$, $(\rho_{4,0}, y_{4,0}) \in \mathring{\mathcal{D}}_2$. This case is covered by the previous theorem.

We conclude with the following.

Theorem 6.3.3. *Let J be a junction with 2 incoming and 2 outgoing roads. A generic equilibrium is stable.*

6.3.2 (AR-2): Maximize the Density

By Theorem 6.2.13, we know that all equilibria in outgoing roads must be in the region $\mathcal{D}_2$. We notice that the instability for the equilibrium for the Riemann problem at J happens when there is a jump in incoming roads from the region $\mathring{\mathcal{D}}_1$ to the region $\mathring{\mathcal{D}}_2$.

We have some possibilities.

1. $(\rho_{i,0}, y_{i,0}) \in \mathring{\mathcal{D}}_1$ for every $i \in \{1, \ldots, n\}$ and $y_{j,0} = \rho_{j,0}$ for some $j \in \{n+1, \ldots, n+m\}$. This implies that $\delta_{i,0} := y_{i,0} - \rho_{i,0}^{\gamma+1} = \sup \Omega_i$ for every $i \in \{1, \ldots, n\}$ and $\delta_{j,0} := y_{j,0} - \rho_{j,0}^{\gamma+1} \leqslant \sup \Omega_j$ for every $j \in \{n+1, \ldots, n+m\}$. Moreover there exists $j_1 \in \{n+1, \ldots, n+m\}$ such that $\delta_{j_1,0} = \sup \Omega_{j_1}$. This means that all the incoming roads and at least one outgoing road give a constraint for the maximization problem (6.2.15). This fact implies that the equilibrium is unstable. Indeed consider an incoming road I_i and a wave of the first family connecting $(\tilde{\rho}_i, \tilde{y}_i)$ with $(\rho_{i,0}, y_{i,0})$ such that the set $\widetilde{\Omega}_i$, defined as in (6.2.10) for the state $(\tilde{\rho}_i, \tilde{y}_i)$, strictly contains Ω_i. There are at least n active constraints, so the point of maximum does not change and, if the perturbation is sufficiently small, then we produce a jump on the road I_i.

2. $(\rho_{i,0}, y_{i,0}) \in \mathring{\mathcal{D}}_1$ for every $i \in \{1, \ldots, n\}$ and $y_{j,0} < \rho_{j,0}$ for every $j \in \{n+1, \ldots, n+m\}$. Define η as

$$\eta := \min_{j \in \{n+1,\ldots,n+m\}} \left\{ \sup \Omega_j - \left(y_j - \rho_j^{\gamma+1} \right) \right\},$$

then, by hypotheses we have that $\eta > 0$. Assume that a wave of the first family on an outgoing road I_j connecting $(\rho_{j,0}, y_{j,0})$ with $(\tilde{\rho}_j, \tilde{y}_j)$ arrives to J. If the perturbation is sufficiently small, then the new set $\tilde{\Omega}_j$ defined as in (6.2.12) with the new state $(\tilde{\rho}_j, \tilde{y}_j)$ satisfies

$$\left| \sup \tilde{\Omega}_j - \sup \Omega_j \right| \leqslant \frac{\eta}{2}$$

and this implies that the maximization problem (6.2.15) remains unchanged. Then only a wave of the second family on I_j connecting $(\hat{\rho}_j, \hat{y}_j)$ with $(\tilde{\rho}_j, \tilde{y}_j)$ is created. Moreover if the perturbation is sufficiently small, then

$$|(\hat{\rho}_j, \hat{y}_j) - (\tilde{\rho}_j, \tilde{y}_j)| \leqslant C\,|(\rho_{j,0}, y_{j,0}) - (\tilde{\rho}_j, \tilde{y}_j)|,$$

where C is a positive constant. Now, suppose that a wave connecting $(\tilde{\rho}_i, \tilde{y}_i)$ with $(\rho_{i,0}, y_{i,0})$ arrives at J. Assume first $(\rho_{i,0}, y_{i,0}) \in \mathring{\mathcal{D}}_1$. If the perturbation is sufficiently small, then:

(a) the new solution of the maximization problem (6.2.15) is given by

$$(\delta_{1,0} := y_{1,0} - \rho_{1,0}^{\gamma+1}, \ldots, \hat{\delta}_i, \ldots, \delta_{n,0} := y_{n,0} - \rho_{n,0}^{\gamma+1})$$

with $\hat{\delta}_i := \tilde{y}_i - \tilde{\rho}_i^{\gamma+1}$ and the final state $(\hat{\rho}_i, \hat{y}_i)$ is equal to $(\tilde{\rho}_i, \tilde{y}_i)$;

(b) the solution

$$(\hat{\delta}_{n+1}, \ldots, \hat{\delta}_{n+m})^T = A \cdot (\delta_{1,0}, \ldots, \hat{\delta}_i, \ldots, \delta_{n,0})^T$$

satisfies $\hat{\delta}_j < \sup \Omega_j$ for every $j \in \{n+1, \ldots, n+m\}$, the final states $(\hat{\rho}_j, \hat{y}_j)$ are such that $\hat{y}_j < \hat{\rho}_j$ for every $j \in \{n+1, \ldots, n+m\}$ (the outgoing roads are not constraints for the maximization problem (6.2.15)) and

$$\sum_{j=n+1}^{n+m} |(\hat{\rho}_j, \hat{y}_j) - (\rho_{j,0}, y_{j,0})| < C\,|(\tilde{\rho}_i, \tilde{y}_i) - (\rho_{i,0}, y_{i,0})|$$

for some C positive constant.

If, on the contrary, $(\rho_{i,0}, y_{i,0})$ is on the curve of maxima, then $\left| \hat{\delta}_i - \delta_{i,0} \right|$ is proportional to the incoming wave, $(\hat{\rho}_i, \hat{y}_i)$ is on the curve of maxima, (b) holds and we conclude similarly. So the equilibrium is stable.

3. $(\rho_{i,0}, y_{i,0}) \in \mathring{\mathcal{D}}_2$ for every $i \in \{1, \ldots, n\}$, and $y_{j,0} = \rho_{j,0}$ for at least n indices $j \in \{n+1, \ldots, n+m\}$. Define

$$\eta := \min_{i \in \{1,\ldots,n\}} \left\{ \sup \Omega_i - \left(y_i - \rho_i^{\gamma+1} \right) \right\}.$$

If from an incoming road I_i a wave connecting $(\tilde{\rho}_i, \tilde{y}_i)$ with $(\rho_{i,0}, y_{i,0})$ arrives at J, then the new set $\tilde{\Omega}_i$, defined as in (6.2.10) for the state $(\tilde{\rho}_i, \tilde{y}_i)$, satisfies

$$\left|\sup \tilde{\Omega}_i - \sup \Omega_i\right| \leqslant \frac{\eta}{2}$$

provided that the perturbation is sufficiently small. Thus the maximization problem (6.2.15) remains unchanged and only a wave of the first family connecting $(\tilde{\rho}_i, \tilde{y}_i)$ with $(\hat{\rho}_i, \hat{y}_i)$ is created. Moreover,

$$|(\hat{\rho}_i, \hat{y}_i) - (\tilde{\rho}_i, \tilde{y}_i)| \leqslant C \left|(\rho_i, y_i) - (\tilde{\rho}_i, \tilde{y}_i)\right|,$$

where C is a positive constant.

A similar case happens if the perturbation is on an outgoing road I_j with $y_{j,0} < \rho_{j,0}$.

Now, consider a wave connecting $(\rho_{j,0}, y_{j,0})$ with $(\tilde{\rho}_j, \tilde{y}_j)$ on an outgoing road I_j with $y_{j,0} = \rho_{j,0}$. For the maximization problem (6.2.15), the active constraints remain the same. Waves are produced only in incoming roads and on outgoing roads that give no active constraints. Then

$$\sum_{i=1}^{n+m} |(\hat{\rho}_i, \hat{y}_i) - (\rho_{i,0}, y_{i,0})| \leqslant C \left|(\hat{\rho}_j, \hat{y}_j) - (\rho_{j,0}, y_{j,0})\right|,$$

where C is a positive constant. Thus the equilibrium is stable.

Putting together all the previous results, we obtain the following.

Theorem 6.3.4. *If* $(\rho_{i,0}, y_{i,0}) \in \mathcal{D}_1$ *for every* $i \in \{1, \ldots, n\}$ *and* $y_{j,0} < \rho_{j,0}$ *for every* $j \in \{n+1, \ldots, n+m\}$*, then the equilibrium is stable.*

If $(\rho_{i,0}, y_{i,0}) \in \mathring{\mathcal{D}}_2$ *for every* $i \in \{1, \ldots, n\}$ *and* $y_{j,0} = \rho_{j,0}$ *for at least* n *outgoing roads, then the equilibrium is stable.*

Consider now the generic case when $m = n = 2$. Generically the states in outgoing roads belong to the region $y < \rho$, hence the outgoing roads are not constraints. Therefore there is only one generic case: the incoming roads are constraints for the maximization problem (6.2.15). So this is a stable equilibrium by Theorem 6.3.4 We deduce the following theorem.

Theorem 6.3.5. *Let* J *be a junction with* 2 *incoming and* 2 *outgoing roads. A generic equilibrium is stable.*

6.3.3 (AR-3): Minimize the Total Variation

Notice that, in this case, the instability for the equilibrium happens when there is a jump in incoming roads from the region $\mathring{\mathcal{D}}_1$ to the region $\mathring{\mathcal{D}}_2$. We have some possibilities.

1. For every index $j \in \{n+1, \dots, n+m\}$, $(\rho_{j,0}, y_{j,0}) \in \mathcal{D} \setminus \{(\rho, y) : \rho = y, \rho \geqslant (\frac{1}{\gamma+1})^{1/\gamma}\}$. Then $(\rho_{i,0}, y_{i,0}) \in \mathcal{D}_1$, $y_{i,0} - \rho_{i,0}^{\gamma+1} = \sup \Omega_i$ for every $i \in \{1, \dots, n\}$ and $y_{j,0} - \rho_{j,0}^{\gamma+1} < \sup \Omega_j$ for every $j \in \{n+1, \dots, n+m\}$. Define

$$\eta := \min_{j \in \{n+1,\dots,n+m\}} \left\{ \sup \Omega_j - \left(y_{j,0} - \rho_{j,0}^{\gamma+1} \right) \right\}.$$

Assume that a wave connecting $(\rho_{j,0}, y_{j,0})$ with $(\tilde{\rho}_j, \tilde{y}_j)$ reaches J. This may happen only if $(\rho_{j,0}, y_{j,0}) \in \mathring{\mathcal{D}}_2$. If the wave is sufficiently small, then

$$\left| \sup \widetilde{\Omega}_j - \sup \Omega_j \right| \leqslant \frac{\eta}{2}$$

which implies that the maximization problem (6.2.15) remains unchanged. Only a wave connecting $(\hat{\rho}_j, \hat{y}_j)$ with $(\tilde{\rho}_j, \tilde{y}_j)$ is created. Moreover, if the perturbation is small, then

$$|(\hat{\rho}_j, \hat{y}_j) - (\tilde{\rho}_j, \tilde{y}_j)| \leqslant C \left| (\rho_{j,0}, y_{j,0}) - (\tilde{\rho}_j, \tilde{y}_j) \right|,$$

with C positive constant. Now, consider a wave connecting $(\tilde{\rho}_i, \tilde{y}_i)$ with $(\rho_{i,0}, y_{i,0})$ on the incoming road I_i. If the perturbation is sufficiently small, then the maximization problem (6.2.15) has the following solution:

$$(\delta_{1,0}, \dots, \tilde{\delta}_i, \dots, \delta_{n,0}),$$

with $\tilde{\delta}_i := \sup \widetilde{\Omega}_i$ and $\delta_{l,0} := y_{l,0} - \rho_{l,0}^{\gamma+1}$. Moreover the fluxes of the density in the outgoing roads change in a continuous way with respect the strength of the perturbation. Thus this equilibrium is stable.

2. $(\rho_{i,0}, y_{i,0}) \in \mathring{\mathcal{D}}_1$ for every $i \in \{1, \dots, n\}$ and $y_{j,0} = \rho_{j,0}$, $(\rho_{j,0}, y_{j,0}) \in \mathcal{D}_2$ for some $j \in \{n+1, \dots, n+m\}$. This is an unstable case. In fact, if a wave on an incoming road I_i reaches J, in such a way the set Ω_i increases, then the maximization problem (6.2.15) admits the same point of maximum (at least one outgoing road is an active constraint) and a jump happens in the incoming road I_i.

3. $(\rho_{i,0}, y_{i,0}) \in \mathring{\mathcal{D}}_2$ for every $i \in \{1, \dots, n\}$ and $y_{j,0} = \rho_{j,0}$, $(\rho_{j,0}, y_{j,0}) \in \mathcal{D}_2$ for at least n indices $j \in \{n+1, \dots, n+m\}$. The active constraints are given by the outgoing roads. For small perturbations these are again the only active constraints. Thus the equilibrium is stable.

Putting together all the previous results we have:

Theorem 6.3.6. *If $(\rho_{i,0}, y_{i,0}) \in \mathcal{D}_1$ for every $i \in \{1, \dots, n\}$ and if, for every $j \in \{n+1, \dots, n+m\}$, $(\rho_{j,0}, y_{j,0}) \in \mathcal{D} \setminus \{(\rho, y) : \rho = y, \rho \geqslant (\frac{1}{\gamma+1})^{1/\gamma}\}$, then the equilibrium is stable.*

If $(\rho_{i,0}, y_{i,0}) \in \mathring{\mathcal{D}}_2$ for every $i \in \{1, \dots, n\}$ and $y_{j,0} = \rho_{j,0}$, $(\rho_{j,0}, y_{j,0}) \in \mathcal{D}_2$ for at least n indices $j \in \{n+1, \dots, n+m\}$, then the equilibrium is stable.

Consider now the generic case when $m = n = 2$. As in the previous subsection, the outgoing roads are not constraints. Therefore there is only one generic case: the incoming roads are constraints for the maximization problem (6.2.15). So this is a stable equilibrium by Theorem 6.3.6 We have the following theorem.

Theorem 6.3.7. *Let J be a junction with 2 incoming and 2 outgoing roads. A generic equilibrium is stable.*

6.4 Existence of Solutions to a Cauchy Problem

This section deals with the existence of solutions to a Cauchy problem in a road network with only one junction J and n incoming and m outgoing roads. Fix a road network with this property and a stable equilibrium $((\rho_{1,0}, y_{1,0}), \ldots, (\rho_{n+m,0}, y_{n+m,0}))$ for the Riemann problem at J for one of the additional rules (AR-1),(AR-2) or (AR-3). Assume the following hypothesis:

(H) there exist $k_1, k_2 \in \{1, 2\}$ such that $(\rho_{i,0}, y_{i,0}) \in \mathring{\mathcal{D}}_{k_1}$ for every $i \in \{1, \ldots, n\}$ and $(\rho_{j,0}, y_{j,0}) \in \mathring{\mathcal{D}}_{k_2}$ for every $j \in \{n+1, \ldots, n+m\}$.

By the analysis of the previous section, the following proposition holds.

Proposition 6.4.1. *There exists a positive constant C such that, if a wave in an incoming road I_i connecting $(\tilde{\rho}_i, \tilde{y}_i)$ with $(\rho_{i,0}, y_{i,0})$ arrives at J and if the wave has sufficiently small total variation, then the solution to the Riemann problem at J $((\hat{\rho}_1, \hat{y}_1), \ldots, (\hat{\rho}_{n+m}, \hat{y}_{n+m}))$ has the following properties:*

1. *if $(\rho_{l,0}, y_{l,0}) \in \mathring{\mathcal{D}}_i$ for some $i \in \{1, 2\}$ and $l \in \{1, \ldots, n+m\}$, then $(\hat{\rho}_l, \hat{y}_l) \in \mathring{\mathcal{D}}_1$;*
2. *we have:*

$$\sum_{l=1, l\neq i}^{n+m} |(\hat{\rho}_l, \hat{y}_l) - (\rho_{l,0}, y_{l,0})| + |(\hat{\rho}_i, \hat{y}_i) - (\tilde{\rho}_i, \tilde{y}_i)| \leqslant C \, |(\tilde{\rho}_i, \tilde{y}_i) - (\rho_{i,0}, y_{i,0})| \, .$$

The same holds for a perturbation on an outgoing road.

Under assumption (H) we can prove the following theorem.

Theorem 6.4.2. *Assume (H), then there exists $\varepsilon > 0$ such that the following holds. For every initial datum*

$$((\rho_{1,0}(x), y_{1,0}(x)), \ldots, (\rho_{n+m,0}(x), y_{n+m,0}(x)))$$

with

$$\|(\rho_{l,0}(x), y_{l,0}(x))\|_{BV} \leqslant \varepsilon$$

and

$$\sup_{x\in(a_l,b_l)} |\rho_{l,0}(x) - \rho_{l,0}| + \sup_{x\in(a_l,b_l)} |y_{l,0}(x) - y_{l,0}| \leqslant \varepsilon$$

for every $l \in \{1, \ldots, n+m\}$*, there exists a solution*

$$((\rho_1(t,x), y_1(t,x)), \ldots, (\rho_{n+m}(t,x), y_{n+m}(t,x))),$$

defined for every $t \geqslant 0$*, such that*

1. $(\rho_l(0,x), y_l(0,x)) = (\rho_{l,0}(x), y_{l,0}(x))$ *for a.e.* $x \in I_l$ *and for every* $l \in \{1, \ldots, n+m\}$*;*
2. $(\rho_l(t,x), y_l(t,x))$ *is an entropic solution to (6.1.1) on each road* I_l*;*
3. *for a.e.* $t > 0$*,*

$$((\rho_1(t,b_1-), y_1(t,b_1-)), \ldots, (\rho_{n+m}(t,a_{n+m}+), y_{n+m}(t,a_{n+m}+)))$$

provides an equilibrium at J*.*

Proof. We consider a wave-front tracking approximate solution on the network; see Chapter 4. Notice that the system is locally rich near the constant solution. So, by Theorem 4.3.11, a wave-front tracking approximate solution exists in the network for every positive time. For every $t > 0$, we denote by (x_k^i, σ_k^i) and (z_l^i, θ_l^i) the positions and strengths in the road I_i of all waves respectively of the first family and of the second family, where k and l belong to some finite sets of indices. For every road I_i, we consider the two functionals

$$V_i(t) := \sum_k |\sigma_k^i| + \sum_l |\theta_l^i|$$

and

$$Q_i(t) := \sum_{z_l^i < x_k^i} |\sigma_k^i \theta_l^i| + \sum_{\sigma_k^i < 0} |\sigma_k^i \sigma_{k'}^i|,$$

which are the classical components of the Glimm functional (notice that the second family is linearly degenerate, hence in the functional Q_i interactions between waves of the second family do not appear). We introduce also a functional $\widetilde{V}$ measuring the strength of waves approaching J. If $i \in \{1, \ldots, n\}$, then define

$$\widetilde{V}_i(t) := \begin{cases} V_i(t), & \text{if } (\rho_{i,0}, y_{i,0}) \in \mathring{\mathcal{D}}_1, \\ \sum_l |\theta_l^i|, & \text{if } (\rho_{i,0}, y_{i,0}) \in \mathring{\mathcal{D}}_2. \end{cases}$$

For an outgoing road I_j, we put

$$\widetilde{V}_j(t) := \begin{cases} 0, & \text{if } (\rho_{j,0}, y_{j,0}) \in \mathring{\mathcal{D}}_1, \\ \sum_k \left|\sigma_k^j\right|, & \text{if } (\rho_{j,0}, y_{j,0}) \in \mathring{\mathcal{D}}_2. \end{cases}$$

Define $V(t) := \sum_{i=1}^{n+m} V_i(t)$, $Q(t) := \sum_{i=1}^{n+m} Q_i(t)$ and $\widetilde{V}(t) := \sum_{i=1}^{n+m} \widetilde{V}_i(t)$. We claim that there exist two positive constants C_1 and C_2 such that the functional

$$\Upsilon(t) := V(t) + C_1 \widetilde{V}(t) + C_2 Q(t)$$

is decreasing in time.

Assuming this, for every $t > 0$,

$$\begin{aligned}\Upsilon(t) &\leqslant \Upsilon(0) = V(0) + C_1 \widetilde{V}(0) + C_2 Q(0) \\ &\leqslant V(0) + C_1 V(0) + C_2 V^2(0)\end{aligned}$$

and, since Υ is equivalent to the total variation as norm, then the total variation of the approximate wave front tracking solution remains bounded for every $t > 0$, hence we have the conclusion by standard compactness arguments.

We prove now that Υ is decreasing in time. Clearly Υ changes only at times where two waves interact or a wave approaches J. If at a time $\tau > 0$ two waves interact in a road I_i, then, by standard estimates (see [19]), we have

$$\begin{aligned}\Delta V_i(\tau) &\leqslant C \cdot \text{ product of strength of waves}, \\ \Delta \widetilde{V}_i(\tau) &\leqslant C \cdot \text{ product of strength of waves}, \\ \Delta Q_i(\tau) &\leqslant -\frac{\text{product of strength of waves}}{2},\end{aligned} \tag{6.4.20}$$

for some $C > 0$, provided that V is sufficiently small. If

$$\frac{C_2}{2} \geqslant C(1 + C_1), \tag{6.4.21}$$

then $\Delta\Upsilon \leqslant 0$ when waves interact in the roads. Consider now an interaction of a wave with J. For simplicity we assume that a wave of the second family (z_l^1, θ_l^1) arrives at J from the incoming road I_1 at time τ. The other cases are completely similar. By Proposition 6.4.1, we have that:

$$\Delta V(\tau) \leqslant C \left|\theta_l^1\right|, \quad \Delta Q(\tau) \leqslant C \left|\theta_l^1\right| V(\tau-),$$

and

$$\Delta \widetilde{V}_1(\tau) = -\left|\theta_k^1\right|, \quad \Delta \widetilde{V}_i(\tau) = 0 \quad \text{for } i \neq 1.$$

Therefore $\Delta \widetilde{V}(\tau) = -\left|\theta_k^1\right|$ and

$$\Delta\Upsilon(\tau) \leqslant C \left|\theta_k^1\right| - C_1 \left|\theta_k^1\right| + C_2 C \left|\theta_k^1\right| V(\tau-).$$

If

$$C_1 \geqslant C + C C_2 V, \tag{6.4.22}$$

then $\Delta\Upsilon \leqslant 0$ when a wave interacts with J.

Fix $C_1 \geqslant C$. Then it is possible to take C_2 satisfying (6.4.21). There exists $\delta > 0$ depending on C_1 and C_2 such that, if $V < \delta$, then (6.4.22) and (6.4.20) hold. As long as $V < \delta$, then Υ is decreasing, thus

$$
\begin{aligned}
V(t) &\leqslant \Upsilon(t) \leqslant \Upsilon(0) = V(0) + C_1 \widetilde{V}(0) + C_2 Q(0) \\
&\leqslant (1 + C_1) V(0) + C_2 V^2(0) \leqslant C_3 \cdot V(0),
\end{aligned}
$$

for some constant $C_3 > 1$. Choosing $\varepsilon = \frac{\delta}{(n+m)C_3}$, we have that $V(0) \leqslant \frac{\delta}{C_3}$ thus $V(t) \leqslant \delta$ for every $t > 0$. So we conclude the proof. □

6.5 Open Problems

Problem 6.5.1. Existence of solutions to a general Cauchy problem. The main difficult in this direction is that the total variation, when there are waves passing arbitrary near to the empty $(0, 0)$, may blow up. Another difficult is that, in the domain $\mathcal{D}$, the Aw-Rascle model is not a rich system.

6.6 Bibliographical Note

The Aw-Rascle model on a network was considered by Garavello and Piccoli [47], by Herty and Rascle [65] and by Klar [76].

A network with the second order phase transition model was considered also by Colombo and Garavello [31].

Appendix: Total Variation of the Flux

In the case of a road network where the Lighthill-Whitham-Richards scalar model is considered in each road, if every junction has exactly 2 incoming and 2 outgoing roads, then an increment of the total variation of the flux can happen only when a wave on an outgoing road interacts with the junction; see Chapter 5. Here the situation is different since there are cases in which the total variation of the flux of the density strictly increases after an interaction of a wave from an incoming road, even if we are considering a junction with 2 incoming and 2 outgoing roads. In fact, consider a junction J with I_1 and I_2 incoming roads and I_3 and I_4 outgoing roads. Moreover suppose $\gamma = 1$ and the matrix A defined by

$$
A = \begin{pmatrix} \frac{1}{3} & \frac{1}{2} \\ \frac{2}{3} & \frac{1}{2} \end{pmatrix}.
$$

Consider the point

$$
(\delta_{1,0}, \delta_{2,0}, \delta_{3,0}, \delta_{4,0}) = (1/8, 1/8, 5/48, 7/48).
$$

It is clear that

$$
\begin{pmatrix} \delta_{3,0} \\ \delta_{4,0} \end{pmatrix} = A \cdot \begin{pmatrix} \delta_{1,0} \\ \delta_{2,0} \end{pmatrix}.
$$

We show that there exists an equilibrium configuration with $\delta_{i,0}$ as density fluxes. In I_1 we consider the point on the curve of maxima

$$(\rho_{1,0}, y_{1,0}) = \left(\frac{1}{2\sqrt{2}}, \frac{1}{4}\right)$$

so that $\Omega_1 = [0, 1/8]$ and $y_{1,0} - \rho_{1,0}^2 = 1/8$. In road I_2 we consider a point $(\rho_{2,0}, y_{2,0})$ such that $y_{2,0} - \rho_{2,0}^2 = 1/8$, $y_{2,0} < 2\rho_{2,0}^2$ and $\frac{1}{8} < \sup \Omega_2$. In road I_3 we consider the point

$$(\rho_{3,0}, y_{3,0}) = \left(\frac{1+\sqrt{\frac{7}{12}}}{2}, \frac{1+\sqrt{\frac{7}{12}}}{2}\right)$$

and so $\frac{5}{48} = y_{3,0} - \rho_{3,0}^2 = \sup \Omega_3$. Finally in I_4 we consider a point $(\rho_{4,0}, y_{4,0})$ such that $y_{4,0} - \rho_{4,0}^2 = \frac{7}{48} < \sup \Omega_4$. Notice that for every additional rule, it is possible to choose $(\rho_{4,0}, y_{4,0})$ such that $((\rho_{1,0}, y_{1,0}), \ldots, (\rho_{4,0}, y_{4,0}))$ is an equilibrium for the Riemann problem at J. For this equilibrium, the active constraints are given by roads I_1 and I_3.

We perturb the equilibrium with a wave of the second family connecting $(\tilde{\rho}_1, \tilde{y}_1)$ with $(\rho_{1,0}, y_{1,0})$ such that the set $\widetilde{\Omega}_1$, defined as in (6.2.10) for the state $(\tilde{\rho}_1, \tilde{y}_1)$, is equal to $[0, 1/8 + \varepsilon]$, where ε is a small positive parameter. This is possible by taking

$$(\tilde{\rho}_1, \tilde{y}_1) = \left(\frac{\frac{1}{4}+2\varepsilon}{\sqrt{\frac{1}{8}+\varepsilon}} - \frac{\sqrt{2}}{4}, \frac{\left(\frac{1}{4}+2\varepsilon\right)^2}{\frac{1}{8}+\varepsilon} - \frac{\sqrt{2}}{4}\cdot\frac{\frac{1}{4}+2\varepsilon}{\sqrt{\frac{1}{8}+\varepsilon}}\right) \in \mathring{\mathcal{D}}_2.$$

The new solution of (6.2.15) is given by

$$(\hat{\delta}_1, \hat{\delta}_2, \hat{\delta}_3, \hat{\delta}_4) = (1/8 + \varepsilon, 1/8 - 2\varepsilon/3, 5/48, 7/48 + \varepsilon/3).$$

Therefore the total variation of the first component of the flux after the interaction is given by

$$\begin{aligned}\left|\hat{\delta}_1 - \delta_1\right| + \left|\hat{\delta}_2 - \delta_{2,0}\right| + \left|\hat{\delta}_3 - \delta_{3,0}\right| + \left|\hat{\delta}_4 - \delta_{4,0}\right| &= \\ = \hat{\delta}_1 - \delta_1 + \frac{2}{3}\varepsilon + \frac{\varepsilon}{3} = \frac{1}{8} - \delta_1 + 2\varepsilon,\end{aligned}$$

where $\delta_1 = y_1 - \rho_1^2 < \hat{\delta}_1$. Instead, the total variation of the first component of the flux before the interaction is given by

$$|\delta_1 - \delta_{1,0}| = \delta_1 - \frac{1}{8},$$

and so an increment of the total variation of the density flux happens, since $\delta_1 < \hat{\delta}_1 = 1/8 + \varepsilon$.

7

Source Destination Model

This chapter deals with an extension of the LWR model on a network including sources and destinations, that are, respectively, areas for which cars start their travels and areas where they end. Thus, on each road, we consider the car density ρ and a vector π describing the traffic types, i.e. the percentages of cars going from a fixed source to a fixed destination. We assume that each car has a precise path inside the network. Such paths determine the behavior at junctions via the coefficients π. For the evolution of the density ρ, we use the fluidodynamic model proposed by Lighthill, Whitham and Richards, while the evolution of π follows a semilinear equation of the type

$$\pi_t + v(\rho)\pi_x = 0. \tag{7.0.1}$$

This is obtained assuming that the velocity of cars depend only on the density and not on the traffic type. The fluxes of the Riemann solver of Chapter 5 are not continuous with respect to the traffic types π. Then we propose a new Riemann solver to overcome this difficulty and prove existence of solutions for Cauchy problems with initial data which are perturbations of equilibria.

7.1 Basic Definitions

In this section we introduce the basic notation for the source-destination model. Let us start with the definition of *SD-road network*.

Definition 7.1.1. *A SD-road network is a 7-tuple*

$$(\mathcal{I}, \mathcal{F}, \mathcal{J}, \mathcal{S}, \mathcal{D}, \mathcal{R}, \mathcal{P})$$

where:

Edges: $\mathcal{I}$ *is a finite collection of intervals, called roads,* $I_i = [a_i, b_i] \subseteq \mathbb{R}$, $i = 1, \ldots, N$*;*

Fluxes: $\mathcal{F}$ *is a finite collection of fluxes* $f^i : [0, \rho^i_{max}] \mapsto \mathbb{R}$;

Junctions: $\mathcal{J}$ *is a finite collection of subsets of* $\{\pm 1, \ldots, \pm N\}$. *If* $j \in J \in \mathcal{J}$, *then the road* $I_{|j|}$ *is crossing at* J *as entering road (i.e. at point* b_i*) if* $j > 0$ *and as exiting road (i.e. at point* a_i*) if* $j < 0$. *For each junction* $J \in \mathcal{J}$, *we indicate by* $\mathrm{Inc}(J)$ *the set of incoming roads, that are* I_i*'s such that* $i \in J$ *and* $i > 0$, *while by* $\mathrm{Out}(J)$ *the set of outgoing roads, that are* I_i*'s such that* $-i \in J$ *and* $i > 0$;

Sources: $\mathcal{S}$ *is a finite subset of* $\{1, \ldots, N\}$, *representing roads connected to traffic sources;*

Destinations: $\mathcal{D}$ *is a finite subset of* $\{1, \ldots, N\}$, *representing roads leading to final destinations;*

Traffic distribution functions: $\mathcal{R}$ *is a finite collection of functions* $r_J : \mathrm{Inc}(J) \times \mathcal{S} \times \mathcal{D} \to \mathrm{Out}(J)$;

Right of way parameters: $\mathcal{P}$ *is a finite collection of vectors* $P_J \in \mathbb{R}^l$ *and* $l = \sharp\,\mathrm{Inc}(J) - 1$.

The previous definition introduces some new concepts: sources, destinations, traffic distribution functions and right of way parameters. The meanings of roads, fluxes, junctions, sources and destinations are clear. Each traffic distribution function r_J indicates the direction at the junction J of traffic that started at source s and has d as final destination. Notice that we need $\mathrm{Inc}(J) \neq \emptyset$ and $\mathrm{Out}(J) \neq \emptyset$ in order r_J to be well defined, and we give additional conditions later to have a suitable network. The right of way parameters determine a level of "importance" at the junctions for incoming roads.

On each road I_i of the network, consider the evolution equation

$$\rho^i_t + f^i(\rho^i)_x = 0. \tag{7.1.2}$$

Hence the datum on the network is given by a finite collection of functions $\rho^i : [0, +\infty[\times I_i \mapsto [0, \rho^i_{max}]$, $i = 1, \ldots, N$. We require entropy-admissible solution to equation (7.1.2); see Chapter 2 and Chapter 5.

At sources and destinations, we assume that boundary data are assigned and the solution is interpreted in the sense of [15].

7.1.1 Traffic Distribution at Junctions

The evolution of car densities ρ^i is thus described on roads and at sources and destinations. To treat the evolution at junctions, we introduce some definitions.

Fix a junction J and assume for notational simplicity that $\mathrm{Inc}(J) = \{1, \ldots, n\}$ and $\mathrm{Out}(J) = \{n+1, \ldots, n+m\}$.

Definition 7.1.2. *A weak solution at the junction* J *is a collection of functions* $\rho^i : [0, +\infty[\times I_i \to [0, \rho^i_{max}]$, $i = 1, \ldots, n+m$, *satisfying*

$$\sum_{l=0}^{n+m} \left(\int_0^{+\infty} \int_{a_l}^{b_l} \left(\rho^l \frac{\partial \varphi_l}{\partial t} + f^l(\rho^l) \frac{\partial \varphi_l}{\partial x} \right) dx dt \right) = 0 \tag{7.1.3}$$

for each $\varphi_1, ..., \varphi_{n+m}$ *smooth having compact support in* $]0, +\infty[\times\mathbb{R}$*, that are also smooth across the junction, i.e.*

$$\varphi_i(\cdot, b_i) = \varphi_j(\cdot, a_j), \qquad \frac{\partial\varphi_i}{\partial x}(\cdot, b_i) = \frac{\partial\varphi_j}{\partial x}(\cdot, a_j),$$

for every $i = 1, \ldots, n$*, and* $j = n+1, \ldots, n+m$*.*

Definition 7.1.3. *A traffic-type function on a road* I_i *is a function*

$$\pi^i : [0, \infty[\times[a_i, b_i] \times \mathcal{S} \times \mathcal{D} \to [0, 1]$$

such that, for every $t \in [0, \infty[$ *and* $x \in [a_i, b_i]$*,*

$$\sum_{s\in\mathcal{S}, d\in\mathcal{D}} \pi^i(t, x, s, d) = 1.$$

Hence $\pi^i(t, x, s, d)$ specifies the amount of the density $\rho^i(t, x)$ that started from source s and is moving towards the final destination d.

Let us now show how a solution at the junction J is constructed using π and r_J. Fix a time t and assume that for all $i \in \mathrm{Inc}(J)$, $s \in \mathcal{S}$ and $d \in \mathcal{D}$, $\pi^i(t, \cdot, s, d)$ admits a limit at the junction J, i.e. left limit at b_i. For $i \in \{1, \ldots, n\}$, $j \in \{n+1, \ldots, n+m\}$, set

$$\alpha_{j,i} = \sum_{s\in\mathcal{S}, d\in\mathcal{D}, r_J(i,s,d)=j} \pi^i(t, b_i-, s, d). \tag{7.1.4}$$

Notice that $\alpha_{j,i}$ is the amount of the density ρ^i that flow towards road I_j. From the definition of traffic-type functions we get:

$$\alpha_{j,i} \in [0, 1], \qquad \sum_{j\in\mathrm{Out}(J)} \alpha_{j,i} = 1.$$

The fluxes $f^j(\rho^j)$ to be consistent with the traffic-type functions must satisfy the following relations:

$$f^j(\rho^j(\cdot, a_j+)) = \sum_{i=1}^{n} \alpha_{j,i} f^i(\rho^i(\cdot, b_i-)), \tag{7.1.5}$$

for each $j = n+1, ..., n+m$. However this is not sufficient to determine a unique solution. Hence we introduce the concept of admissible weak solution, using also the right of way parameters.

Definition 7.1.4. *Let* $\rho = (\rho^1, \ldots, \rho^{n+m})$ *be such that* $\rho^i(t, \cdot)$ *is of bounded variation for every* $t \geqslant 0$*. Then* ρ *is an* admissible weak solution *at the junction* J *related to the matrix* A *and to the right of way parameters* $P_J = (p_1, \ldots, p_{n-1}) \in \mathbb{R}_+^{n-1}$ *if and only if the following properties hold:*

(i) ρ *is a weak solution at the junction* J*;*

(ii) $f^j(\rho^j(\cdot, a_j+)) = \sum_{i=1}^{n} \alpha_{j,i} f^i(\rho^i(\cdot, b_i-))$*, for each* $j = n+1, ..., n+m$*;*

(iii) the functional

$$c_2 \sum_{i=1}^{n} f^i(\rho^i(\cdot, b_i-)) - c_1 \left[\mathrm{dist}((f^1(\rho^1(\cdot, b_1-)), \ldots, f^n(\rho^n(\cdot, b_n-))), r)\right]^2$$

is maximum subject to (ii), where c_1 *and* c_2 *are strictly positive constants, and* $\mathrm{dist}(\cdot, r)$ *denotes the euclidean distance in* $\mathbb{R}^n$ *from the line* r*, which is given by*

$$\begin{cases} \gamma_n = p_1 \gamma_1, \\ \vdots \\ \gamma_n = p_{n-1} \gamma_{n-1}. \end{cases}$$

Remark 7.1.5. Conditions (i) and (ii) do not suffice to solve Riemann problems at junctions in a unique way. Thus one has to impose some modelling assumptions. A further possibility could be to impose

$$(f^1(\rho^1(\cdot, b_1-)), \ldots, f^n(\rho^n(\cdot, b_n-))) \in r$$

and then to maximize with respect to a free parameter. Unfortunately this choice is not realistic. In fact, let us consider a simple junction with 2 incoming roads and 1 outgoing road. If the outgoing road and one incoming road are empty, then just the origin in the (γ_1, γ_2) plane satisfies the previous conditions. Therefore for this model, no car passes through the junction, which is clearly anti intuitive.

If we substitute property (iii) of Definition 7.1.4 with the following

(iii') $\sum_{i=1}^{n} f^i(\rho^i(\cdot, b_i-))$ is maximum subject to (ii),

i.e. we are considering the same solution as in Chapter 5, then uniqueness is not granted. In fact the condition (C) in Chapter 5, necessary for uniqueness, may fail. Moreover, with this choice, the solution for the fluxes does not depend in a continuous way by the matrix A, which on the contrary is the case for our quadratic maximization.

7.1.2 Evolution Equations for Traffic-Type Functions

We assume the followings. Inside each road I_i, cars move at some averaged speed v^i that depends on the local density ρ^i. Moreover v^i is independent from π. In this case the flux function is given by:

$$f^i(\rho^i) = v^i(\rho^i)\, \rho^i.$$

If $x(t)$ denotes a trajectory of a car inside the road I_i, then we get

$$\pi^i(t, x(t), s, d) = \text{const.}$$

Taking the total differential with respect to the time, we deduce the semilinear equation:

$$\partial_t \pi^i(t, x, s, d) + \partial_x \pi^i(t, x, s, d) \cdot v^i(\rho^i(t, x)) = 0. \tag{7.1.6}$$

This equation is coupled with equation (7.1.2). More precisely on road I_i equation (7.1.6) depends on the solution of (7.1.2), while in turn at junctions the values of π^i determine the traffic distribution on outgoing roads as explained in the previous section.

7.1.3 Admissible Networks and Solutions

Let us now give some admissibility conditions on the network.

Definition 7.1.6. *A network $(\mathcal{I}, \mathcal{F}, \mathcal{J}, \mathcal{S}, \mathcal{D}, \mathcal{R}, \mathcal{P})$ is admissible if the following conditions hold.*

1. *Each $f^i \in \mathcal{F}$ is given by $f^i(\rho^i) = v^i(\rho^i)\rho^i$, where v^i, the velocity field, is smooth, strictly decreasing, and $v^i(\rho^i_{max}) = 0$.*
2. *Each $f^i \in \mathcal{F}$ is a smooth strictly concave function with*
$$f^i(0) = f^i(\rho^i_{max}) = 0.$$
Thus there exists a unique $\sigma^i \in]0, \rho^i_{max}[$ global maximum with $(f^i)'(\sigma^i) = 0$.
3. *Each $i \in \{\pm 1, \ldots, \pm N\}$ belongs to at most one $J \in \mathcal{J}$.*
4. *For each $i \in \{1, \ldots, N\}$, exactly one of the following cases happens:*
   a) there exists $J \in \mathcal{J}$ such that $i \in J \cap \mathcal{S}$;
   b) there exists $J \in \mathcal{J}$ such that $-i \in J$ and $i \in \mathcal{D}$;
   c) there exist $J, J' \in \mathcal{J}$ such that $i \in J$ and $-i \in J'$.
5. *$\mathcal{S} \cap \mathcal{D} = \emptyset$.*
6. *For every $J \in \mathcal{J}$, $P_J \in]0, +\infty[^l$, where $l = \sharp\,\mathrm{Inc}(J) - 1$.*
7. *For every $s \in \mathcal{S}$ and $d \in \mathcal{D}$, the functions r_J determine a unique path, that is a finite sequence of roads-junctions $I_{i_1}, J_{l_1}, \cdots, I_{i_k}, J_{l_k}, I_{i_{k+1}}$ such that*
   a) $i_1 = s$, $i_{k+1} = d$;
   b) $i_h \in \mathrm{Inc}(J_{l_h})$ for every $h \in \{1, \ldots, k\}$;
   c) $J_{l_h} \neq J_{l_{h'}}$ for every $h, h' \in \{1, \ldots, k\}$, $h \neq h'$;
   d) $r_{J_{l_h}}(i_h, s, d) = i_{h+1}$ for every $h \in \{1, \ldots, k\}$.

Remark 7.1.7. The first two conditions in the previous definition are needed to have consistency to the model. In particular we assume that the speed of cars is decreasing with respect to the quantity of cars in roads and is equal to zero when the density is maximum; hence the flux must be zero if the density is maximum. For some traffic model v^i may explode at 0. Moreover the concavity condition of the flux implies that the speed of ρ-waves may vary in the interval $[f^{i\prime}(\rho^i_{max}), f^{i\prime}(0)]$.

Conditions 3. and 4. imply that each road is connected at least with one junction and that each road either can be connected with a source and a junction, or can be connected with a destination and a junction or finally can be connected with two junctions.

Condition 5. ensures to avoid path triviality.

Condition 6. gives admissible weights for priorities of incoming roads of junctions.

Finally condition 7. implies existence and uniqueness for paths connecting each source and each destination.

Given an admissible network we have to specify how to define a solution in relation to sources, destinations and junctions.

Definition 7.1.8. *Consider an admissible network* $(\mathcal{I}, \mathcal{F}, \mathcal{J}, \mathcal{S}, \mathcal{D}, \mathcal{R}, \mathcal{P})$. *A set of Initial-Boundary Conditions (briefly IBC) is given assigning measurable functions* $\bar{\rho}^i : I_i \to [0, \rho^i_{max}]$, $\bar{\pi}^i : [a_i, b_i] \times \mathcal{S} \times \mathcal{D} \to [0, 1]$, $i = 1, \dots, N$ *and measurable functions* $\psi^i : [0, +\infty[\to [0, \rho^i_{max}]$, $i \in \mathcal{S} \cup \mathcal{D}$ *with the property that for every* $s \in \mathcal{S}$ *and* $d \in \mathcal{D}$, $\bar{\pi}^i(\cdot, s, d) = 0$ *whenever* I_i *does not belong to the path for the source* s *and destination* d *defined at the point 7 of Definition 7.1.6*

Definition 7.1.9. *Consider an admissible network* $(\mathcal{I}, \mathcal{F}, \mathcal{J}, \mathcal{S}, \mathcal{D}, \mathcal{R}, \mathcal{P})$ *and a set of IBC. A set of functions* $\rho = (\rho^1, \dots, \rho^N)$ *with* $\rho^i : [0, +\infty[\times I_i \to [0, \rho^i_{max}]$ *continuous as functions from* $[0, +\infty[$ *into* L^1, *and* $\Pi = (\pi^1, \dots, \pi^N)$ *with* $\pi^i : [0, +\infty[\times I_i \times \mathcal{S} \times \mathcal{D} \to [0, 1]$, *continuous as functions from* $[0, +\infty[$ *into* L^1 *for every* $s \in \mathcal{S}, d \in \mathcal{D}$, *is an admissible solution if the following holds. Each* ρ^i *is a weak entropic solution to (7.1.2) on* I_i, $\rho^i(0, x) = \bar{\rho}^i(x)$ *for almost every* $x \in [a_i, b_i]$, $\rho^i(t, a_i) = \psi^i(t)$ *if* $i \in \mathcal{S}$ *and* $\rho^i(t, b_i) = \psi^i(t)$ *if* $i \in \mathcal{D}$ *in the sense of [15]. Each* π^i *is a weak solution to the corresponding equation (7.1.6). Finally at each junction* ρ *is a weak solution and is an admissible weak solution for* Π *in case of bounded variation.*

Regarding sources and destinations, the treatment of boundary data in the sense of [15] can be done in the same way as in [3, 5]. Thus we treat the construction of solutions only inside the network.

7.2 The Riemann Problem

Consider a junction J in which there are n roads with incoming traffic and m roads with outgoing traffic. For simplicity we indicate by

$$(t, x) \in \mathbb{R}_+ \times I_i \mapsto \rho^i(t, x) \in [0, \rho^i_{max}], \quad i = 1, \dots, n, \tag{7.2.7}$$

the densities of the cars on the road with incoming traffic and

$$(t,x) \in \mathbb{R}_+ \times I_j \mapsto \rho^j(t,x) \in [0, \rho^j_{max}], \quad j = n+1, ..., n+m, \tag{7.2.8}$$

those on the roads with outgoing traffic.

Without loss of generality, we assume that the fluxes f^i, f^j ($i \in \{1, \ldots, n\}$, $j \in \{n+1, \ldots, n+m\}$) are all the same and we indicate them with f. Hence we assume $\rho^i_{max} = \rho^j_{max} = 1$ and we have $\sigma = \sigma^i = \sigma^j$ for every $i \in \{1, \ldots, n\}$ and $j \in \{n+1, \ldots, n+m\}$.

Theorem 7.2.1. *Let $(\mathcal{I}, \mathcal{F}, \mathcal{J}, \mathcal{S}, \mathcal{D}, \mathcal{R}, \mathcal{P})$ be an admissible network and J a junction with n incoming roads and m outgoing ones. For every initial data $\rho_{1,0}, ..., \rho_{n+m,0} \in [0,1]$, and $\pi^{1,s,d}, \ldots, \pi^{n+m,s,d} \in [0,1]$ ($s \in \mathcal{S}$, $d \in \mathcal{D}$), there exist a unique admissible weak solution, in the sense of Definition 7.1.4, $\rho = \left(\rho^1, ..., \rho^{n+m}\right)$ and traffic-type functions $(\pi^1(\cdot,\cdot,s,d), \ldots, \pi^{n+m}(\cdot,\cdot,s,d))$ at the junction J such that*

$$\rho^1(0,\cdot) \equiv \rho_{1,0}, \ldots, \rho^{n+m}(0,\cdot) \equiv \rho_{n+m,0},$$
$$\pi^1(0,\cdot,s,d) = \pi^{1,s,d}, \ldots, \pi^{n+m}(0,\cdot,s,d) = \pi^{n+m,s,d}, \quad (s \in \mathcal{S}, d \in \mathcal{D}).$$

Moreover for every $i \in \{1, \ldots, n\}$ there exists $\hat{\rho}_i$ such that the solution for the density in I_i is given by the wave generated by the Riemann problem with initial data $(\rho_{i,0}, \hat{\rho}_i)$, while for every $j \in \{n+1, \ldots, n+m\}$ there exists $\hat{\rho}_j$ such that the solution for the density in I_j is given by the wave generated by the Riemann problem with initial data $(\hat{\rho}_j, \rho_{j,0})$.

Moreover $\pi^i(t,\cdot,s,d) = \pi^{i,s,d}$ for every $t \geqslant 0$, $i \in \{1, \ldots, n\}$, $s \in \mathcal{S}$, $d \in \mathcal{D}$ and

$$\pi^j(t, a_j+, s, d) = \frac{\sum_{r_J(i,s,d)=j} \pi^{i,s,d} f(\hat{\rho}_i)}{f(\hat{\rho}_j)} \tag{7.2.9}$$

for every $t \geqslant 0$, $j \in \{n+1, \ldots, n+m\}$, $s \in \mathcal{S}$, $d \in \mathcal{D}$.

Proof. In $\mathbb{R}^n$, define r to be the linear subspace

$$\begin{cases} \gamma_n = p_1\gamma_1, \\ \vdots \\ \gamma_n = p_{n-1}\gamma_{n-1}, \end{cases} \tag{7.2.10}$$

which is clearly a line in $\mathbb{R}^n$ by Definition 7.1.6 Consider the function $E : \mathbb{R}^n \to \mathbb{R}$ defined by

$$E(\gamma_1, \ldots, \gamma_n) = c_2 \sum_{i=1}^n \gamma_i - c_1[\mathrm{dist}((\gamma_1, \ldots, \gamma_n), r)]^2, \tag{7.2.11}$$

where $\mathrm{dist}(\cdot, r)$ denotes the usual euclidean distance in $\mathbb{R}^n$ from r. Moreover, as in Chapter 5, we define the sets

$$\Omega_i := \begin{cases} [0, f(\rho_{i,0})], & \text{if } 0 \leqslant \rho_{i,0} \leqslant \sigma, \\ [0, f(\sigma)], & \text{if } \sigma \leqslant \rho_{i,0} \leqslant 1, \end{cases} \qquad i = 1, \ldots, n,$$

$$\Omega_j := \begin{cases} [0, f(\sigma)], & \text{if } 0 \leqslant \rho_{j,0} \leqslant \sigma, \\ [0, f(\rho_{j,0})], & \text{if } \sigma \leqslant \rho_{j,0} \leqslant 1, \end{cases} \qquad j = n+1, \dots, n+m,$$

and

$$\Omega := \left\{ (\gamma_1, \dots, \gamma_n) \in \Omega_1 \times \cdots \times \Omega_n : A \cdot (\gamma_1, \dots, \gamma_n)^T \in \Omega_{n+1} \times \cdots \times \Omega_{n+m} \right\}, \tag{7.2.12}$$

where the entries $\alpha_{j,i}$ of the matrix A are given by (7.1.4). The set Ω is clearly convex, compact and not empty. To define the solution to the Riemann problem at J we have to solve the maximization problem

$$\sup_{(\gamma_1, \dots, \gamma_n) \in \Omega} E(\gamma_1, \dots, \gamma_n). \tag{7.2.13}$$

Since E is a continuous function and Ω is a compact set, the maximization problem admits a solution. Let us suppose that $(\hat{\gamma}_1, \dots, \hat{\gamma}_n) \in \Omega$ and $(\tilde{\gamma}_1, \dots, \tilde{\gamma}_n) \in \Omega$ satisfy

$$E(\hat{\gamma}_1, \dots, \hat{\gamma}_n) = E(\tilde{\gamma}_1, \dots, \tilde{\gamma}_n) = \sup_{(\gamma_1, \dots, \gamma_n) \in \Omega} E(\gamma_1, \dots, \gamma_n). \tag{7.2.14}$$

The Hessian matrix of E is clearly equal to the Hessian matrix of the function

$$-c_1 \left[\text{dist}((\gamma_1, \dots, \gamma_n), r) \right]^2, \tag{7.2.15}$$

since the term $c_2 \sum_{i=1}^n \gamma_i$ is linear. If $(\nu_1, \dots, \nu_n)$ is an orthogonal system where the first coordinate has the same direction of r, then the Hessian matrix of (7.2.15) has the form

$$-2c_1 \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & I_{n-1} & \\ 0 & & & \end{pmatrix}, \tag{7.2.16}$$

where I_{n-1} is the $(n-1) \times (n-1)$ identity matrix. Clearly (7.2.16) is a semi-negative definite matrix. This analysis shows that if $z_1, z_2 \in \mathbb{R}^n$, $z_1 \neq z_2$ and the line through z_1 and z_2 is not parallel to r, then

$$E(\lambda z_1 + (1-\lambda) z_2) > \lambda E(z_1) + (1-\lambda) E(z_2) \tag{7.2.17}$$

for every $\lambda \in (0, 1)$.

Suppose by contradiction that $(\hat{\gamma}_1, \dots, \hat{\gamma}_n) \neq (\tilde{\gamma}_1, \dots, \tilde{\gamma}_n)$. If the line through these two points is parallel to r, then (7.2.11) and (7.2.14) give

$$c_2 \sum_{i=1}^n (\hat{\gamma}_i - \tilde{\gamma}_i) = 0$$

and so $(\hat{\gamma}_1, \dots, \hat{\gamma}_n) = (\tilde{\gamma}_1, \dots, \tilde{\gamma}_n)$ since r intersects the hyperplane $\sum_{i=1}^n \gamma_i = 0$ just in the origin. Therefore (7.2.14) implies that the line through the points $(\hat{\gamma}_1, \dots, \hat{\gamma}_n)$ and $(\tilde{\gamma}_1, \dots, \tilde{\gamma}_n)$ is not parallel to r and so (7.2.17) gives

$$\begin{aligned} &E(\lambda(\hat{\gamma}_1,\ldots,\hat{\gamma}_n)+(1-\lambda)(\tilde{\gamma}_1,\ldots,\tilde{\gamma}_n)) \\ &> \lambda E(\hat{\gamma}_1,\ldots,\hat{\gamma}_n)+(1-\lambda)E(\tilde{\gamma}_1,\ldots,\tilde{\gamma}_n) \\ &= \sup_{(\gamma_1,\ldots,\gamma_n)\in\Omega} E(\gamma_1,\ldots,\gamma_n) \end{aligned}$$

for every $\lambda \in (0,1)$, which is a contradiction. Therefore $(\hat{\gamma}_1,\ldots,\hat{\gamma}_n)$ is equal to $(\tilde{\gamma}_1,\ldots,\tilde{\gamma}_n)$, i.e. the point of maximum of E is unique.

For every $i \in \{1,...,n\}$, we choose $\hat{\rho}_i \in [0,1]$ such that

$$f(\hat{\rho}_i)=\hat{\gamma}_i, \quad \hat{\rho}_i \in \begin{cases} \{\rho_{i,0}\}\cup]\tau(\rho_{i,0}),1], & \text{if } 0 \leqslant \rho_{i,0} \leqslant \sigma, \\ [\sigma,1], & \text{if } \sigma \leqslant \rho_{i,0} \leqslant 1. \end{cases}$$

Assumptions on f imply that $\hat{\rho}_i$ exists and is unique. Let

$$\hat{\gamma}_j \doteq \sum_{i=1}^{n} \alpha_{ji}\hat{\gamma}_i, \qquad j=n+1,...,n+m,$$

and $\hat{\rho}_j \in [0,1]$ be such that

$$f(\hat{\rho}_j)=\hat{\gamma}_j, \quad \hat{\rho}_j \in \begin{cases} [0,\sigma], & \text{if } 0 \leqslant \rho_{j,0} \leqslant \sigma, \\ \{\rho_{j,0}\}\cup[0,\tau(\rho_{j,0})[, & \text{if } \sigma \leqslant \rho_{j,0} \leqslant 1. \end{cases}$$

Since $(\hat{\gamma}_1,...,\hat{\gamma}_n)\in\Omega$, $\hat{\rho}_j$ exists and is unique for every $j \in \{n+1,\ldots,n+m\}$. Solving the Riemann Problem on each road, the first claim is proved.

The speeds v^i of the traffic-type functions are positive; hence

$$\pi^i(t,\cdot,s,d)=\pi^i(0,\cdot,s,d)=\pi^{i,s,d}$$

for every $t \geqslant 0$, $i \in \{1,\ldots,n\}$, $s \in \mathcal{S}$, $d \in \mathcal{D}$. Finally, if $t \geqslant 0$, $j \in \{n+1,\ldots,n+m\}$, $s \in \mathcal{S}$, $d \in \mathcal{D}$, then $\pi^j(t,a_j+,s,d)$ is the percentage of $\rho^j(t,a_j+)$ which corresponds to cars going from the source s to the destination d. Therefore it corresponds to the ratio

$$\frac{\sum_{i=1}^{n} \pi^{i,s,d} f(\hat{\rho}_i)}{f(\hat{\rho}_j)},$$

that is the quantity of cars going from s to d over the global amount of cars in I_j. □

Remark 7.2.2. Notice that it may happen that a single road is the unique constraint for the maximization problem (7.2.13). This is due to the fact that the level curves of the function E are paraboloids and the solution to the maximization problem can be given by the tangent point to a level curve of E with a face of the boundary of Ω.

Remark 7.2.3. The solution to the Riemann problem at junctions is different from that introduced in Chapter 5. Indeed, the solution of the Riemann problem given in Chapter 5 requires the additional condition (C), necessary for uniqueness, and that junctions have not two incoming and one outgoing roads.

The choice of Chapter 5 is not good for this model for the following reasons.

1. Since the matrix A is given by formula (7.1.4), to satisfy condition (C) we would need to impose some assumptions on the traffic-type functions, which are very technical and not meaningful from modeling point of view.
2. The solution to the Riemann problem at the junction does not depend in a continuous way from the coefficients π. Indeed a small variation on the coefficients of A may create a big variation in the solution of a Riemann problem.

On the contrary, the solution given here does not require condition (C) on the matrix A and satisfies the property that small changes in the coefficients of A produces small changes in the solution for the fluxes.

Remark 7.2.4. Clearly the solution to the Riemann problem at a junction involves also the values of traffic-type functions, since they determine the entries of the matrix A. More precisely, only the values of the traffic-type functions in incoming roads influence the matrix A, while the values of the traffic-type functions in outgoing roads are determined by the solution of the Riemann problem for the density.

Remark 7.2.5. Condition 7 in Definition 7.1.6 imply that, for every Initial-Boundary Condition and $s \in \mathcal{S}$, $d \in \mathcal{D}$, at most one $i \in \{1, \dots, n\}$ is such that $\pi^{i,s,d} \neq 0$. Therefore (7.2.9) can be rewritten in the following way: for every $s \in \mathcal{S}$, $d \in \mathcal{D}$, $j \in \{n+1, \dots, n+m\}$, there exists $i \in \{1, \dots, n\}$ such that

$$\pi^j(t, a_j+, s, d) = \frac{\pi^{i,s,d} f(\hat{\rho}_i)}{f(\hat{\rho}_j)}. \tag{7.2.18}$$

7.2.1 Junctions with two Incoming and two Outgoing Roads

Let us introduce the definition of equilibrium for a Riemann problem at a junction and the definition of genericity for an equilibrium.

Definition 7.2.6. *Let J be a junction with n incoming roads, say $I_1, \dots, I_n$, and m outgoing roads, say $I_{n+1}, \dots, I_{n+m}$, and let A be a fixed distributional matrix at J.*

We say that $(\rho_1, \dots, \rho_{n+m})$ is an equilibrium at J if the solution to the Riemann problem at J $(\hat{\rho}_1, \dots, \hat{\rho}_{n+m})$ with the initial data $(\rho_1, \dots, \rho_{n+m})$ coincides with $(\rho_1, \dots, \rho_{n+m})$.

We say that $(\rho_1, \dots, \rho_{n+m})$ is a generic equilibrium to the Riemann problem at J if the following two conditions are satisfied:

1. *the set Ω defined in (7.2.12) is different from $\{(0, \dots, 0)\}$;*
2. *either the solution to the maximization problem (7.2.13) belongs to the interior of one faces of Ω or belongs to a vertex of Ω generated by the intersection of exactly n faces of Ω.*

Fix a junction J with two incoming roads I_1, I_2 and two outgoing ones I_3 and I_4, consider a distributional matrix

$$A = \begin{pmatrix} \alpha & \beta \\ 1-\alpha & 1-\beta \end{pmatrix} \tag{7.2.19}$$

and suppose that $\alpha > \beta$ and $0 < p_1 < 1$. We study in detail equilibria (i.e. constant solutions) for the Riemann problem at J when a single road is the unique active constraint for the maximization problem (7.2.13). In this case the function $E : \mathbb{R}^2 \to \mathbb{R}$, defined by

$$E(\gamma_1, \gamma_2) = c_2(\gamma_1 + \gamma_2) - c_1[\mathrm{dist}((\gamma_1, \gamma_2), r)]^2,$$

can be explicitly rewritten in the form

$$E(\gamma_1, \gamma_2) = -\frac{c_1}{1+p_1^2}(\gamma_2 - p_1\gamma_1)^2 + c_2(\gamma_1 + \gamma_2). \tag{7.2.20}$$

Dini's implicit function theorem implies that the maximum $(\bar{\gamma}_1, \bar{\gamma}_2)$ of E over Ω satisfies:

1. $\bar{\gamma}_2 = p_1\bar{\gamma}_1 + \frac{(1+p_1^2)c_2}{2c_1}$, provided the road I_1 is the only active constraint;
2. $\bar{\gamma}_2 = p_1\bar{\gamma}_1 - \frac{(1+p_1^2)c_2}{2c_1 p_1}$, provided the road I_2 is the only active constraint;
3. $\bar{\gamma}_2 = p_1\bar{\gamma}_1 + \frac{(1+p_1^2)c_2(\alpha-\beta)}{2c_1(\alpha+p_1\beta)}$, provided the road I_3 is the only active constraint;
4. $\bar{\gamma}_2 = p_1\bar{\gamma}_1 + \frac{(1+p_1^2)c_2(\beta-\alpha)}{2c_1(p_1+1-\alpha-p_1\beta)}$, provided the road I_4 is the only active constraint.

Moreover the axis for the parabolas which are level curves of E in the (γ_1, γ_2) coordinates is given by the line

$$\gamma_2 = p_1\gamma_1 + \frac{c_2(1-p_1)}{2c_1}. \tag{7.2.21}$$

Table 7.1 describes all the possible generic equilibria both for the Riemann solver introduced in this chapter and for that in Chapter 5. Notice that equilibria with only one active constraint are not admissible for the Riemann solver of Chapter 5. Moreover some other types of equilibria (3 and 7 in table 7.1) are not admissible for the Riemann solver of Chapter 5.

If we impose conditions on $f(\sigma)$, then not all cases can happen as shown by next results.

Lemma 7.2.7. *If* $f(\sigma) < \frac{c_2(1+p_1^2)(\alpha-\beta)}{2c_1p_1(p_1+1-p_1\beta-\alpha)}$, *the equilibria 5, 7 and 10 in table 7.1 can not happen.*

If $f(\sigma) < \frac{c_2(1+p_1^2)(\alpha-\beta)}{2c_1(p_1\beta+\alpha)}$, *the equilibria 1, 3 and 8 in table 7.1 can not happen.*

	Active constraints	new Riemann solver	Riemann solver of Ch. 5
1	I_1	yes	no
2	I_1 and I_2	yes	yes
3	I_1 and I_3	yes	no
4	I_1 and I_4	yes	yes
5	I_2	yes	no
6	I_2 and I_3	yes	yes
7	I_2 and I_4	yes	no
8	I_3	yes	no
9	I_3 and I_4	yes	yes
10	I_4	yes	no

Table 7.1. equilibria for the Riemann solver introduced in this paper and for the Riemann solver introduced in Chapter 5 when $\alpha > \beta$.

Proof. If $f(\sigma) < \frac{c_2(1+p_1^2)(\alpha-\beta)}{2c_1p_1(p_1+1-p_1\beta-\alpha)}$, then the region between the lines

$$\gamma_2 = p_1\gamma_1 - \frac{(1+p_1^2)c_2}{2c_1p_1}$$

and

$$\gamma_2 = p_1\gamma_1 + \frac{(1+p_1^2)c_2(\beta-\alpha)}{2c_1(p_1+1-\alpha-p_1\beta)}$$

in the (γ_1, γ_2) plane do not intersect Ω and so the first statement holds. In the same way the second statement is proved. □

Corollary 7.2.8. *If* $f(\sigma) < \min\left\{\frac{c_2(1+p_1^2)(\alpha-\beta)}{2c_1p_1(p_1+1-p_1\beta-\alpha)}, \frac{c_2(1+p_1^2)(\alpha-\beta)}{2c_1(p_1\beta+\alpha)}\right\}$, *then the Riemann solver introduced in this chapter and that introduced in Chapter 5 have the same types of equilibria.*

Finally, for a simple network consisting of a single junction, we get the following proposition.

Proposition 7.2.9. *Let us consider an admissible SD-road network with just one junction J, two incoming roads I_1 and I_2 and two outgoing roads I_3 and I_4. Assume that in incoming roads there are not Π-waves, so that the matrix A for the junction J is fixed and given by (7.2.19). If $\alpha > \beta$ and* $f(\sigma) < \min\left\{\frac{c_2(1+p_1^2)(\alpha-\beta)}{2c_1p_1(p_1+1-p_1\beta-\alpha)}, \frac{c_2(1+p_1^2)(\alpha-\beta)}{2c_1(p_1\beta+\alpha)}\right\}$, *then all estimates in Section 5.3 for waves interacting with J hold. Hence for every positive time $T > 0$, an entropic solution on $[0, T]$ exists on the network.*

Proof. By Corollary 7.2.8, we know that the Riemann solvers introduced in this chapter and in Chapter 5 have the same kinds of equilibrium. A deeper analysis shows that an arbitrary initial datum $(\rho_{1,0}, \rho_{2,0}, \rho_{3,0}, \rho_{4,0})$ for the density at J produces the same solution $(\hat{\rho}_1, \hat{\rho}_2, \hat{\rho}_3, \hat{\rho}_4)$ for the both Riemann solvers. In fact the set Ω, defined in (7.2.12) is clearly the same for the two Riemann solvers. Moreover each maximization procedure implies that the maximum is on the boundary of Ω. The maximum, by hypotheses, can be only at a vertex of Ω and the vertex must be the same, since either the roads I_1 and I_2 or the roads I_1 and I_4 or the roads I_2 and I_3 or the roads I_3 and I_4 can be the active constraints, as shown in Table 7.1. The estimates in Section 5.3 depend only on the solution of the Riemann problem and not on the Riemann solver used. Therefore we conclude. □

7.3 Wave-Front Tracking Algorithm

In this section a wave-front tracking algorithm is given for admissible solutions (ρ, Π) in the sense of Definition 7.1.9.

For every $s \in \mathcal{S}$ and $d \in \mathcal{D}$, along each road we need to solve the system

$$\begin{cases} \rho^i_t(t,x) + f^i(\rho^i(t,x))_x = 0, \\ \pi^i_t(t,x,s,d) + v^i(\rho^i(t,x))\pi^i_x(t,x,s,d) = 0. \end{cases} \tag{7.3.22}$$

First one takes piecewise constant approximations, in L^1, ρ^i_n, π^i_n, of the initial data $\bar{\rho}^i$, $\bar{\pi}^i$.

Then we construct a solution for the density solving all the Riemann problems until an interaction between two ρ-waves or between a ρ-wave with a junction. Rarefactions are approximated by rarefaction shock fans always inserting the value σ^i when possible. The speed of a rarefaction shock is set to be the value of $(f^i)'$ at its left endpoint, with the exception that every rarefaction shock with endpoint σ^i has zero speed. Then we construct the solution for the traffic-type functions on the same interval of times. If an interaction of a Π-wave with a junction occurs, then we consider the new distribution matrix at the junction and we recalculate the solution for the density until the first interaction time. Repeating this procedure inductively, we are able to construct a wave-front tracking approximate solution.

To achieve the construction one needs estimates on the number of waves and on the total variation of the solution.

The bound on the number of waves and interactions is obtained as in Chapter 5. The estimate of the total variation is the more delicate issue and is based on some approximation procedures and on basic interaction estimates, shown in the next section.

7.4 Basic Estimates of Interactions

Let us consider an admissible SD-road network $(\mathcal{I}, \mathcal{F}, \mathcal{J}, \mathcal{S}, \mathcal{D}, \mathcal{R}, \mathcal{P})$. Without loss of generality, we assume that $\rho^i_{max} = 1$ and $f^i = f$ for every road of the network. Hence $\sigma^i = \sigma$ for every $i \in \{1, \ldots, n\}$. Moreover in this section we do the following assumption:

(A1) every junction $J \in \mathcal{J}$ has at most two incoming roads and at most two outgoing roads.

Let us consider an equilibrium $(\bar{\rho}, \bar{\Pi})$ on the whole network, that is an admissible solution constant in time.

Definition 7.4.1. *Let J be a junction and let us consider an equilibrium at J. We say that the equilibrium is of the first type if there are exactly two active constraints for the maximization problem (7.2.13) and the corresponding hyperplanes are not tangent to a level curve of E at the equilibrium.*

We say that the equilibrium is of the second type if there is exactly one active constraint for the maximization problem (7.2.13).

We also assume for the rest of the section:

(A2) for every $J \in \mathcal{J}$, the equilibrium is generic.

Remark 7.4.2. A generic equilibrium for the Riemann problem at a junction is either of the first type or of the second type. Other types of equilibria are not generic, because there is at least one active constraint redundant for the maximization problem (7.2.13).

For example, consider a junction J with two incoming roads I_1, I_2 and two outgoing roads I_3, I_4, for which the matrix A is given by

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

and $0 < p_1 < 1$. Let $(\rho_1, \rho_2, \rho_3, \rho_4)$ be an equilibrium such that

$$0 < \rho_1 < \sigma, \quad \sigma < \rho_2 < 1, \quad \sigma < \rho_3 < 1, \quad 0 < \rho_4 < \sigma,$$
$$f(\rho_1) = \frac{1}{2}, \quad f(\rho_2) = \frac{p_1}{2}, \quad f(\rho_3) = f(\rho_4) = \frac{1 + p_1}{2}.$$

In this case the roads I_1 and I_3 are active constraint. Also $(\tilde{\rho}_1, \rho_2, \rho_3, \rho_4)$ with $\sigma < \tilde{\rho}_1 < 1$, $f(\tilde{\rho}_1) = \frac{1}{2}$ is an equilibrium, but in the second case only the road I_3 is an active constraint. So the first equilibrium is not generic, while the second one is.

Our aim is to prove an existence result for a solution (ρ, Π) in the case of a small perturbation of the equilibrium $(\bar{\rho}, \bar{\Pi})$.

We have to consider the following types of interactions:

T1. interactions of ρ-waves with ρ-waves on roads;
T2. interactions of ρ-waves with Π-waves on roads;
T3. interactions of Π-waves with Π-waves on roads;
T4. interactions of ρ-waves with junctions;
T5. interactions of Π-waves with junctions.

Interactions of type T1 are classical and the total variation of the density decreases. Interactions of type T3 can not happen since Π-waves travel with speed depending only on the value of ρ.

Remark 7.4.3. Hypothesis (A2) is fundamental in the next analysis, since it permits to reduce the number of events at junctions and moreover since it excludes the possibility that an outgoing road becomes saturate.

Hypothesis (A2) can be relaxed, but can not be totally eliminated. In fact, if an outgoing road becomes saturated, then some of the next estimates are false.

7.4.1 Interactions of Type T2

Let us consider a road I_i. First we note that the characteristic speed of a density is smaller than the speed of a Π-wave, as next lemma shows.

Lemma 7.4.4. *Let $\rho \in [0,1]$ be a density and let $\lambda(\rho)$ be its characteristic speed. Then $\lambda(\rho) \leqslant v(\rho)$ and the equality holds if and only if $\rho = 0$.*

Proof. By definition, the speed v is strictly decreasing with respect to the density ρ and the flux f is given by $f(\rho) = \rho v(\rho)$. This implies that

$$\lambda(\rho) = f'(\rho) = v(\rho) + \rho v'(\rho) \leqslant v(\rho). \tag{7.4.23}$$

Clearly, if $\rho = 0$, then $\lambda(0) = v(0)$, while if $\rho > 0$, then $\lambda(\rho) < v(\rho)$. □

Lemma 7.4.5. *Let us consider a shock wave connecting ρ^- and ρ^+. Then:*

1. *$\lambda(\rho^-, \rho^+) < v(\rho^-)$;*
2. *$\lambda(\rho^-, \rho^+) \leqslant v(\rho^+)$ and the equality holds if and only if $\rho^- = 0$.*

Proof. We have $\rho^- < \rho^+$ and so $v(\rho^-) > v(\rho^+)$. Thus the first inequality is a direct consequence of the second one. Moreover, the speed of a shock wave is given by the Rankine-Hugoniot condition

$$\lambda(\rho^-, \rho^+) = \frac{f(\rho^+) - f(\rho^-)}{\rho^+ - \rho^-}. \tag{7.4.24}$$

Since $f(\rho) = \rho v(\rho)$, we have that

$$\lambda(\rho^-, \rho^+) \leqslant v(\rho^+) \tag{7.4.25}$$

if and only if

$$\rho^+ v(\rho^+) - \rho^- v(\rho^-) \leqslant \rho^+ v(\rho^+) - \rho^- v(\rho^+), \tag{7.4.26}$$

which is satisfied if and only if

$$\rho^- v(\rho^-) \geqslant \rho^- v(\rho^+). \tag{7.4.27}$$

The last inequality is clearly true. Notice that if $\rho^- \neq 0$ all the previous inequalities are strict inequalities. □

Remark 7.4.6. If we have a shock wave (ρ^-, ρ^+) with $\rho^- > 0$, then by the previous lemma a Π-wave can cross the ρ-wave from the left, since $v(\rho^-) > v(\rho^+) > \lambda(\rho^-, \rho^+)$, that is the speed of the Π-wave when $\rho = \rho^-$ is strictly greater than the speed of the Π-wave when $\rho = \rho^+$, which is strictly greater than the speed of the ρ-wave given by the Rankine-Hugoniot condition; see figure 7.1.

If instead $\rho^- = 0$, then a Π-wave can interact with the ρ-wave since $v(0) > \lambda(\rho^-, \rho^+)$, but then the discontinuity in Π travels with the same speed of the ρ-wave; see figure 7.2.

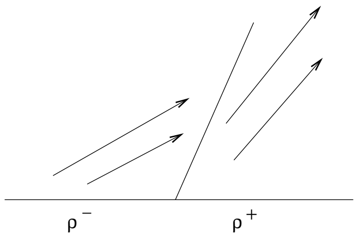


Fig. 7.1. shock wave with $\rho^- > 0$. The speed of the Π-waves is described by the arrows.

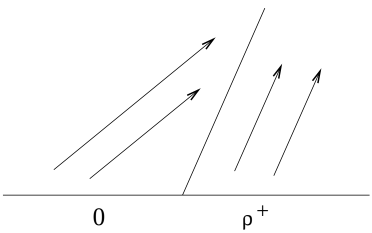


Fig. 7.2. shock wave with $\rho^- = 0$. The speed of the Π-waves is described by the arrows.

Lemma 7.4.7. *Let us consider a rarefaction shock fan connecting ρ^- and ρ^+. Then $v(\rho^+) > v(\rho^-) > f'(\rho^-)$.*

Proof. We have $\rho^- > \rho^+$ and so $v(\rho^-) < v(\rho^+)$. Moreover, $v(\rho^-) > f'(\rho^-)$ if and only if $v(\rho^-) > v(\rho^-) + \rho^- v'(\rho^-)$ and the last inequality is clearly true. □

Remark 7.4.8. If we consider a rarefaction shock fan (ρ^-, ρ^+), then the previous lemma shows that a Π-wave can cross the ρ-wave, since $v(\rho^+)$ and $v(\rho^-)$ are both strictly greater than the speed $f'(\rho^-)$ of the rarefaction shock fan; see figure 7.3.

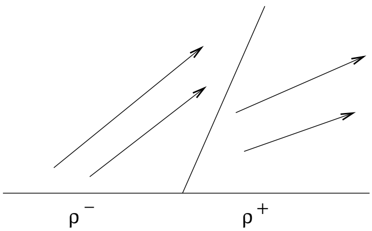


Fig. 7.3. rarefaction shock fan. The speed of the Π-waves is described by the arrows.

Remark 7.4.9. In principle, it is reasonable that the speed of a rarefaction fan can be chosen in the interval $[f'(\rho^-), f'(\rho^+)]$. Once we choose $f'(\rho^-)$ as the speed of a rarefaction fan, Lemma 7.4.7 ensures that the speeds of Π-waves when $\rho = \rho^-$ or $\rho = \rho^+$ are faster than the speed of the rarefaction fan.

If we choose another value $\lambda \in [f'(\rho^-), f'(\rho^+)]$ for the speed of the rarefaction fan, then it may be happen that $v(\rho^-) < \lambda \leqslant v(\rho^+)$ and this creates a problem to construct a wave front-tracking approximate solution for π; see figure 7.4.

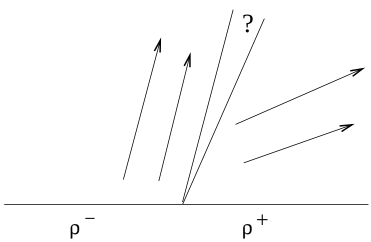


Fig. 7.4. rarefaction shock fan with speed $\lambda > v(\rho^-)$. How to define a wave front-tracking approximate solution for the traffic distribution functions?

Putting together the previous lemmas we obtain the following result.

Proposition 7.4.10. *An interaction of a ρ-wave with a Π-wave can happen only if the Π-wave interacts from the left respect to the ρ-wave. Moreover if* *this happens, then the ρ-wave does not change, while the Π-wave changes only its speed.*

7.4.2 Interactions of Type T4

We consider interactions of ρ-waves with junctions. In general these interactions produce an increment of the total variation of the flux and of the density in all the roads and a variation of the values of traffic-type functions on outgoing roads. As in Section 5.3, we can control the total variation of the flux. Indeed we have the following.

Lemma 7.4.11. *Let J be a junction with at most two incoming roads and two outgoing ones. Let us suppose that a ρ-wave approaches the junction J and assume (A2). If the total variation of the ρ-wave is sufficiently small, then there exists $C > 0$, depending only on the values of the traffic-type functions on incoming roads, such that the total variation of the flux after the wave approaches J is bounded by C times the flux variation of the interacting wave, i.e.*

$$Tot.Var._f^+ \leqslant C\, Tot.Var._f^- .$$

Proof. If the equilibrium at J is of the first type, then the conclusion follows directly from the proof of Lemma 5.3.7

Therefore assume that the equilibrium is of the second type. In this case only one road is an active constraint for the Riemann problem at J. If the total variation of the interacting wave is sufficiently small, then the wave modifies the equilibrium at J if and only if it arrives from the road which is the active constraint, and the equilibrium type does not change, i.e. the constraint remains the same after the interaction. We consider only the case where the first incoming road I_1 is the active constraint, the other cases being similar. We denote by $(\gamma_{1,0}, \ldots, \gamma_{4,0})$ the fluxes of the equilibrium at J, by γ_1 the flux of the interacting wave and by $(\hat{\gamma}_1, \ldots, \hat{\gamma}_4)$ the fluxes of the new equilibrium for the Riemann problem at J. As in subsection 7.2.1, we note that

$$\gamma_{2,0} = p_1\gamma_{1,0} + \frac{(1+p_1^2)c_2}{2c_1}, \quad \hat{\gamma}_2 = p_1\hat{\gamma}_1 + \frac{(1+p_1^2)c_2}{2c_1}, \quad \gamma_1 = \hat{\gamma}_1. \qquad (7.4.28)$$

Thus

$$\text{Tot.Var.}_f^- = |\gamma_1 - \gamma_{1,0}|$$

and

$$\text{Tot.Var.}_f^+ = |\hat{\gamma}_2 - \gamma_{2,0}| + |\hat{\gamma}_3 - \gamma_{3,0}| + |\hat{\gamma}_4 - \gamma_{4,0}| .$$

If $\gamma_1 < \gamma_{1,0}$, then $\hat{\gamma}_2 < \gamma_{2,0}$, $\hat{\gamma}_3 < \gamma_{3,0}$ and $\hat{\gamma}_4 < \gamma_{4,0}$. Therefore

$$\begin{aligned}\text{Tot.Var.}_f^+ &= (\gamma_{2,0} - \hat{\gamma}_2) + (\gamma_{3,0} - \hat{\gamma}_3) + (\gamma_{4,0} - \hat{\gamma}_4)\\ &= (\gamma_{2,0} - \hat{\gamma}_2) + \alpha_{3,1}(\gamma_{1,0} - \gamma_1) + \alpha_{3,2}(\gamma_{2,0} - \hat{\gamma}_2)\\ &\quad +(1-\alpha_{3,1})(\gamma_{1,0} - \gamma_1) + (1-\alpha_{3,2})(\gamma_{2,0} - \hat{\gamma}_2),\end{aligned}$$

since $\gamma_3 = \alpha_{3,1}\gamma_1 + \alpha_{3,2}\gamma_2$ and $\gamma_4 = (1-\alpha_{3,1})\gamma_1 + (1-\alpha_{3,2})\gamma_2$. Using (7.4.28) we conclude that

$$\begin{aligned}\mathrm{Tot.Var.}_f^+ &= 2(\gamma_{2,0} - \hat{\gamma}_2) + (\gamma_{1,0} - \gamma_1) = (1+2p_1)(\gamma_{1,0} - \gamma_1) \\ &= (1+2p_1)\mathrm{Tot.Var.}_f^- .\end{aligned}$$

If $\gamma_1 > \gamma_{1,0}$, then the same calculation shows that

$$\mathrm{Tot.Var.}_f^+ = (1+2p_1)(\gamma_1 - \gamma_{1,0}) = (1+2p_1)\mathrm{Tot.Var.}_f^- .$$

This concludes the proof. □

Remark 7.4.12. Notice that the total variation of the flux can increase also when a wave interacts with a junction from an incoming road even if we start from an equilibrium of the first type. In fact, let us consider a junction J with two incoming roads (I_1 and I_2) and two outgoing ones (I_3 and I_4) and we suppose that the active constraints are given by the roads I_1 and I_3. If a wave interacts with J from I_1, changes the equilibrium configuration and the active constraints remain I_1 and I_3, then the total variation of the flux after the interaction is given by

$$\mathrm{Tot.Var.}_f^+ = |\hat{\gamma}_2 - \gamma_{2,0}| + |\hat{\gamma}_4 - \gamma_{4,0}| ,$$

where $\gamma_{i,0}$ and $\hat{\gamma}_i$ ($i \in \{1,2,3,4\}$) are the fluxes of the equilibrium respectively before and after the interaction. Since the active constraints before and after the interaction remain I_1 and I_3, we have that

$$\alpha_{3,1}\gamma_{1,0} + \alpha_{3,2}\gamma_{2,0} = \alpha_{3,1}\hat{\gamma}_1 + \alpha_{3,2}\hat{\gamma}_2$$

and so

$$\gamma_{2,0} - \hat{\gamma}_2 = \frac{\alpha_{3,1}}{\alpha_{3,2}}(\hat{\gamma}_1 - \gamma_{1,0}).$$

Moreover

$$\gamma_{4,0} = (1-\alpha_{3,1})\gamma_{1,0} + (1-\alpha_{3,2})\gamma_{2,0}, \quad \hat{\gamma}_4 = (1-\alpha_{3,1})\hat{\gamma}_1 + (1-\alpha_{3,2})\hat{\gamma}_2,$$

which implies

$$\gamma_{4,0} - \hat{\gamma}_4 = (1-\alpha_{3,1})(\gamma_{1,0} - \hat{\gamma}_1) + (1-\alpha_{3,2})(\gamma_{2,0} - \hat{\gamma}_2).$$

Thus

$$\mathrm{Tot.Var.}_f^+ = \frac{\alpha_{3,1} + |\alpha_{3,2} - \alpha_{3,1}|}{\alpha_{3,2}}\mathrm{Tot.Var.}_f^- .$$

The conclusion follows from the fact that the coefficient

$$\frac{\alpha_{3,1} + |\alpha_{3,2} - \alpha_{3,1}|}{\alpha_{3,2}} = \begin{cases} 1, & \text{if } \alpha_{3,2} \geqslant \alpha_{3,1}, \\ \frac{2\alpha_{3,1} - \alpha_{3,2}}{\alpha_{3,2}} > 1, & \text{if } \alpha_{3,2} < \alpha_{3,1}. \end{cases}$$

Lemma 7.4.13. *Let J be a junction with at most two incoming roads and two outgoing ones and assume (A2). Suppose that a ρ-wave approaches the junction J. If the total variation of the ρ-wave is sufficiently small, then there exists $C > 0$ such that the variation of the traffic-type functions in outgoing roads is bounded by C times the flux variation of the interacting wave, i.e.*

$$Tot.Var._{\Pi}^{+} \leqslant C\, Tot.Var._{f}^{-}.$$

Proof. Fix a source $s \in \mathcal{S}$ and a destination $d \in \mathcal{D}$. We denote by $\pi_{i,0}$ and $\hat{\pi}_i$ ($i \in \{1,2,3,4\}$) the values of the traffic-type functions for s and d at J, respectively, before and after the interaction of the ρ-wave with J.

If $\pi_{1,0} = \pi_{2,0} = 0$, then clearly $\pi_{3,0} = \pi_{4,0} = 0$ and $\hat{\pi}_3 = \hat{\pi}_4 = 0$ and so we conclude.

Otherwise, since the path for each car is unique, we may assume $\pi_{1,0} \neq 0$, $\pi_{2,0} = 0$, $\pi_{3,0} \neq 0$ and $\pi_{4,0} = 0$. Since the speed of the traffic-type functions is positive, then $\hat{\pi}_1 = \pi_{1,0}$ and $\hat{\pi}_2 = 0$. Moreover we have

$$\pi_{3,0} = \pi_{1,0}\frac{\gamma_{1,0}}{\gamma_{3,0}}, \quad \hat{\pi}_3 = \pi_{1,0}\frac{\hat{\gamma}_1}{\hat{\gamma}_3}, \quad \hat{\pi}_4 = 0,$$

where $\gamma_{i,0}$ and $\hat{\gamma}_i$ denote the flux in road I_i respectively before and after the interaction. Assumption (A2) implies that $\gamma_{3,0} \neq 0$. If the ρ-wave has sufficiently small total variation, then $\hat{\gamma}_3 \neq 0$ and

$$|\pi_{3,0} - \hat{\pi}_3| = \pi_{1,0}\left|\frac{\gamma_{1,0}\hat{\gamma}_3 - \hat{\gamma}_1\gamma_{3,0}}{\hat{\gamma}_3\gamma_{3,0}}\right| \leqslant C\gamma_{1,0}\,|\hat{\gamma}_3 - \gamma_{3,0}| + C\gamma_{3,0}\,|\hat{\gamma}_1 - \gamma_{1,0}|$$

for a suitable constant $C > 0$. Lemma 7.4.11 permits to conclude. □

Remark 7.4.14. The previous lemmas are proved under the assumption that the BV norm of the ρ-wave is such that the type of the equilibrium does not change. This excludes some realistic situations, for example that where an outgoing road becomes closed, due to the increment of the flow in the incoming roads.

7.4.3 Interactions of Type T5

We consider interactions of Π-waves with junctions. Since Π-waves have always positive speed, they can interact with the junction only from an incoming road.

Lemma 7.4.15. *Let us consider a junction J and a Π-wave on an incoming road I_i interacting with J. If A is the distributional matrix for J, whose entries are given by (7.1.4), then the interaction of the Π-wave with J modifies only the i-th column of A. Moreover the variation of the i-th column is bounded by the Π-wave variation.*

Proof. For each $s \in \mathcal{S}$ and $d \in \mathcal{D}$, we denote by $\pi_i^{s,d}$ and $\pi_{i,0}^{s,d}$, respectively, the left and the right states of the Π-wave. Moreover, for every $j \in \{3,4\}$, we denote with $\alpha_{j,i}^-$ and $\alpha_{j,i}^+$, respectively, the entries of the matrix A before and after the interaction of the Π-wave with J. By (7.1.4), it is clear that, if $l \neq i$, then the entries $\alpha_{j,l}$ are not modified. For $l = i$, we have

$$\left|\alpha_{j,i}^+ - \alpha_{j,i}^-\right| \leqslant \sum_{s\in\mathcal{S},d\in\mathcal{D},r_J(i,s,d)=j} \left|\pi_i^{s,d} - \pi_{i,0}^{s,d}\right|.$$

This completes the proof. □

Remark 7.4.16. The previous lemma can be generalized to junctions with n incoming roads and m outgoing ones following the same proof.

Lemma 7.4.17. *Let us consider a junction J and a Π-wave on an incoming road I_i interacting with J. If the total variation of the Π-wave is sufficiently small, then there exists $C > 0$ such that the variation of the fluxes is bounded by C times the Π-wave variation, i.e.*

$$Tot.Var._f^+ \leqslant C\, Tot.Var._\Pi^-.$$

Proof. We consider the case of two incoming and outgoing roads, the other cases being similar. Let us consider first the case of an equilibrium of the first type, i.e. there are exactly two roads that are active constraints for the Riemann problem at J. If the active constraints are given by the incoming roads I_1 and I_2, then the sets Ω_1, Ω_2, Ω_3, Ω_4 and Ω, defined in the proof of Theorem 7.2.1, are given by

$$\Omega_1 = [0, f(\rho_{1,0})], \quad \Omega_2 = [0, f(\rho_{2,0})], \quad \Omega_3 = \Omega_4 = [0, f(\sigma)],$$
$$\Omega = \left\{(\gamma_1,\gamma_2) \in \Omega_1 \times \Omega_2 : A(\gamma_1,\gamma_2)^T \in \Omega_3 \times \Omega_4\right\},$$

where $(\rho_{1,0}, \rho_{2,0}, \rho_{3,0}, \rho_{4,0})$ is the equilibrium for the density at J. Since the Π-wave is sufficiently small, then it does not affect the set Ω. In fact, the perturbation of the Π-wave slightly modifies the constraints given by the lines

$$\alpha_{3,1}\gamma_1 + \alpha_{3,2}\gamma_2 = f(\sigma) \quad \text{and} \quad \alpha_{4,1}\gamma_1 + \alpha_{4,2}\gamma_2 = f(\sigma).$$

These constraints remain non active since

$$\alpha_{3,1}f(\rho_{1,0}) + \alpha_{3,2}f(\rho_{2,0}) < f(\sigma) \quad \text{and} \quad \alpha_{4,1}f(\rho_{1,0}) + \alpha_{4,2}f(\rho_{2,0}) < f(\sigma);$$

see figure 7.5. Hence the solution for the fluxes of the new Riemann problem at J is given by

$$(f(\rho_{1,0}), f(\rho_{2,0}), \alpha_{3,1}^+ f(\rho_{1,0}) + \alpha_{3,2}f(\rho_{2,0}), \alpha_{4,1}^+ f(\rho_{1,0}) + \alpha_{4,2}f(\rho_{2,0})).$$

Therefore ρ-waves appear on I_3 and I_4 and the variation of the fluxes (and of the density) is proportional to the variation of the matrix A.

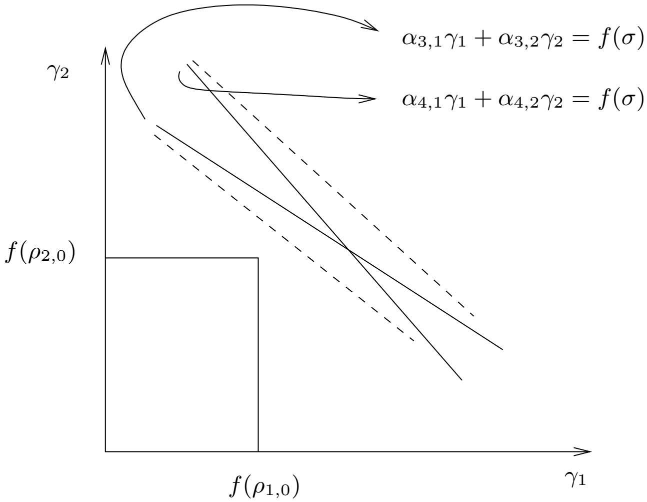


Fig. 7.5. the equilibrium when I_1 and I_2 are active constraints.

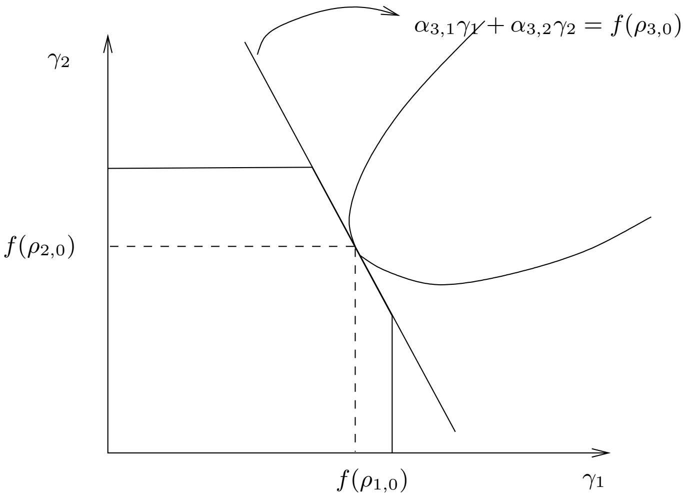


Fig. 7.6. the equilibrium when I_3 is the only active constraint.

If the active constraints are one incoming road and one outgoing road or two outgoing roads, then the conclusion follows in an analogous way.

Let us consider now the case of an equilibrium of the second type. If an incoming road is the active constraint, then the equilibrium does not change for incoming roads and the conclusion is as before. So we suppose that an outgoing road, say I_3, is the active constraint. It means that

$$\Omega_1 = \Omega_2 = \Omega_4 = [0, f(\sigma)], \quad \Omega_3 = [0, f(\rho_{3,0})];$$

see figure 7.6. The change in the matrix A affects the lines in $\mathbb{R}^2$

$$\alpha_{3,1}\gamma_1 + \alpha_{3,2}\gamma_2 = f(\rho_{3,0}), \quad \alpha_{4,1}\gamma_1 + \alpha_{4,2}\gamma_2 = f(\sigma).$$

The new maximum point $(\hat{\gamma}_1, \hat{\gamma}_2)$ for the function E, defined in (7.2.13), belongs to the line

$$\alpha_{3,1}^{+}\gamma_1 + \alpha_{3,2}\gamma_2 = f(\rho_{3,0}),$$

if the perturbation is sufficiently small. In fact, from the shape of the level curves of E, we deduce that $(\hat{\gamma}_1, \hat{\gamma}_2)$ is the tangent point of the previous line to a level curve of E. In our case the function E is given by

$$E(\gamma_1, \gamma_2) = \frac{1}{1 + p_1^2}(\gamma_2 - p_1\gamma_1)^2 - \gamma_1 - \gamma_2.$$

If we denote by m the angular coefficient of

$$\alpha_{3,1}^{+}\gamma_1 + \alpha_{3,2}\gamma_2 = f(\rho_{3,0}),$$

i.e. $m = -\frac{\alpha_{3,1}^{+}}{\alpha_{3,2}}$, then the solution $(\hat{\gamma}_1, \hat{\gamma}_2)$ of the new Riemann problem for the fluxes in the incoming roads is given by the solution of the following system:

$$\begin{cases} \gamma_2 - p_1\gamma_1 = \frac{1+p_1^2}{2}\frac{1+m}{m-p_1}, \\ \alpha_{3,1}^{+}\gamma_1 + \alpha_{3,2}\gamma_2 = f(\rho_{3,0}), \end{cases}$$

where the first equation is the locus of the maximum point for E when the only active constraint is given by I_3, while the second equation gives the expression of the active constraint. This permits to conclude. □

7.5 Perturbations of an equilibrium

We have the following theorem.

Theorem 7.5.1. *Let us consider an admissible SD-road network $(\mathcal{I}, \mathcal{F}, \mathcal{J}, \mathcal{S}, \mathcal{D}, \mathcal{R}, \mathcal{P})$ and assume (A1), (A2) given in the previous section. Let $(\bar{\rho}, \bar{\Pi})$ be an equilibrium on the whole network. For every $T > 0$ there exists $\varepsilon > 0$ such that the following holds. For every perturbation $(\tilde{\rho}, \tilde{\Pi})$ of the equilibrium with*

$$\|\tilde{\rho}\|_{BV} \leqslant \varepsilon, \quad \left\|\tilde{\Pi}\right\|_{BV} \leqslant \varepsilon, \tag{7.5.29}$$

and

$$\|\tilde{\rho} - \bar{\rho}\|_{\infty} + \left\|\tilde{\Pi} - \bar{\Pi}\right\|_{\infty} \leqslant \varepsilon, \tag{7.5.30}$$

there exists an admissible solution (ρ, Π) defined for every $t \in [0, T[$ such that at time $t = 0$ coincides with $(\tilde{\rho}, \tilde{\Pi})$.

Proof. Let $\delta = \min_{I_i}(b_i - a_i)$, $\hat{\lambda} = \max\{f'(0), -f'(1)\}$ and $N = \left[\frac{T\hat{\lambda}}{\delta}\right] + 1$, where the brackets stands for the integer part. For every $\nu \in \mathbb{N}$, let $\hat{\rho}_\nu$, $\hat{\Pi}_\nu$ be two piecewise constant sequences approximating the initial conditions $\tilde{\rho}(0, \cdot)$, $\tilde{\Pi}(0, \cdot)$ in BV-norm. We denote with ρ^*_ν, Π^*_ν an approximate wave-front tracking solution on $[0, T]$ such that $\rho^*_\nu(0, \cdot) = \hat{\rho}_\nu(\cdot)$ and $\Pi^*_\nu(0, \cdot) = \hat{\Pi}_\nu(\cdot)$. For every interaction of a wave with a junction we have the estimates of Lemmas 7.4.11, 7.4.13, 7.4.17, if the strength of the waves are bounded by some $\bar{\varepsilon} > 0$. Taking $\varepsilon = \frac{\bar{\varepsilon}}{NC}$ we get:

$$\text{Tot.Var.}\Pi^*_\nu(t, \cdot) \leqslant NC\varepsilon = \bar{\varepsilon},$$

and

$$\text{Tot.Var.}f(\rho^*_\nu(t, \cdot)) \leqslant NC\varepsilon = \bar{\varepsilon}$$

for every $t \in [0, T]$.

Theorem 5.3.10 implies that there exists ρ^* such that $\rho^*_\nu \to \rho^*$ strongly in L^1_{loc}, at least by extracting a subsequence. Moreover, by Helly theorem, there exists Π^* such that $\Pi^*_\nu \to \Pi^*$ in L^1_{loc}, at least by extracting a subsequence. We complete the proof with standard arguments. □

7.6 Open Problems

Problem 7.6.1. Prove existence of solutions to Cauchy problems in the whole network when the initial datum is not a small perturbation of a generic equilibrium. The main difficulty in this direction is the following: for interactions of waves with junctions it is not possible to bound the variation for the traffic-type functions by a constant times the flux variation of the interacting wave. More precisely Lemma 7.4.13 does not hold in the general case.

Problem 7.6.2. Prove existence of solutions to Cauchy problems in the whole network when the initial datum is a small perturbation of a generic equilibrium, but in some junctions the number of incoming roads or outgoing ones is bigger than or equal to 3.

Problem 7.6.3. Prove or disprove continuous dependence of the solution to Cauchy problems from the initial datum. As in the LWR case, the Lipschitz continuous dependence on the initial datum does not hold in general. What about the continuous dependence or the Lipschitz continuous dependence in special cases?

7.7 Bibliographical Note

The idea of sources and destinations is customarily used in transportation sciences and was already proposed, for instance, in 1965 in a paper by Edie, Gazis, Helly, Herman and Rothery; see [43]. The first paper considering a fluidodynamic model on a road network with sources and destinations was [46]: all the subject of this chapter is based on such paper.

8

An Example of Traffic Regulation: Circles vs Lights

In this chapter we consider the following traffic regulation problem:

(P) when constructing a junction, with a given traffic flux expected, is it preferable a traffic light or a circle?

More precisely, we assume that drivers arriving at the junction distribute on the outgoing roads according to some known coefficients and our aim is to understand which solution performs better from the point of view of total amount of cars going through the junction. Then we can use the L-W-R model on networks with time varying traffic distribution coefficients introduced in Chapter 5.

We determine the asymptotic fluxes first in case of low incoming traffic. In this situation the circle seems to perform better than the light. Then, for heavy incoming traffic, we analyze the probability of the junction to get stuck in dependence of the various parameters of the problem. Notice that if the circle is blocked, with cars bumper to bumper, then all the incoming traffic would be stopped. However, we show that this can be avoided, in most situations, regulating the right of way parameters of the circle. As intuition may suggest, if traffic inside the circle has strong priority w.r.t. the incoming one, then a traffic jam may occur only for very heavy traffic loads. The complete analysis requires shock wave tracing, see Section 8.3.

Then we briefly discuss the case of multi lane traffic circle, where we assume the presence of ramps for incoming traffic. We prove that this situation is equivalent to the case of traffic inside the circle yielding to incoming traffic. As a consequence of the previous analysis, this type of traffic circle performs well only for low traffic.

We conclude the chapter providing a comparison between traffic lights and traffic circles. In general one can see that, in case of low incoming flux, the traffic light is preferable only if there is a necessity of regulating outgoing fluxes. For heavy traffic the right of way parameters of traffic circle can be set so to avoid a complete jam. In conclusion, **it seems preferable to use traffic circles any time the setting of right of way parameters permits**

to avoid a stuck situation, possibly adding some light at crossings with incoming roads that should work either in case of very heavy traffic or to control outgoing fluxes.

8.1 Flux Control for Traffic Lights

In this section we investigate the behavior of traffic flux for a simple network formed by a single junction J with two incoming road I_1 and I_2 and two outgoing roads I_3 and I_4. We refer to Figure 8.1.

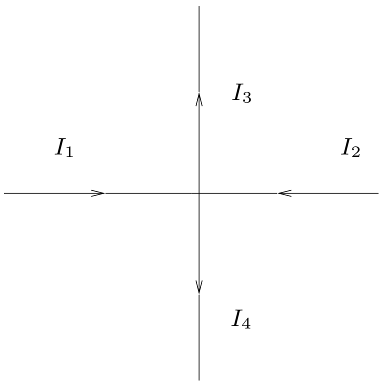


Fig. 8.1. The junction J with the traffic light.

Red and green lights are represented by time varying coefficients $\alpha_{j,i}$. In fact, if green is for road I_1, then $\alpha_{3,2} = \alpha_{4,2} = 0$ and if green is for road I_2, then $\alpha_{3,1} = \alpha_{4,1} = 0$.

8.1.1 Notations and Position of the Control Problem

For simplicity, we assume that $f(\rho) = k\rho(1-\rho)$, where k is a positive constant. Therefore, $\sigma = \frac{1}{2}$ and $\rho_{max} = 1$.

At the traffic light, roads I_1 and I_2 are called the incoming roads (IR) and roads I_3 and I_4 the outgoing roads (OR) (see Figure 8.1). On the incoming roads, the densities of cars arriving at the intersections are respectively equal to $\bar{\rho}_1$ and $\bar{\rho}_2$. Let Δ_g and Δ_r be the times for which the traffic light is green and red for the road I_1. We assume that at $t = 0$, the light is green for I_1. We set

$$X_i = 1 - 2\bar{\rho}_i, \quad \text{for } i = 1, 2, \quad \eta = \frac{\Delta_g}{\Delta_g + \Delta_r}, \quad \alpha = \alpha_{13}, \quad \beta = \alpha_{23}. \tag{8.1.1}$$

Moreover, up to relabelling all the roads, we may assume with no loss of generality, that

$$\alpha < \beta, \quad \bar{\rho}_1 < \bar{\rho}_2. \tag{8.1.2}$$

It should be noticed that most of the results stated in this section usually assume strict inequalities, as in equations (8.1.1) and (8.1.2). However, it is clear that such results extend in the cases of equalities, the latter being limits of the cases of inequalities.

It is clear that the control parameter of the problem is η. Moreover, on each incoming road, the effect of the traffic light can be qualitatively described as follows. The evolution of the car density is provided by a boundary condition at $x = 0$ defined as a piecewise constant periodic function of the time whose period is $\Delta_g + \Delta_r$ and values $\bar{\rho}_i$ (cars are flowing) and ρ_{max} (cars are stopped). The latter situation will generate a backward shock wave along the incoming road and it is desirable that these shock waves do not propagate too far since they correspond to a stuck traffic along the incoming road.

Therefore, we want to choose (if possible) the control parameter η so that

(a) backward shock waves created on the incoming roads stay bounded;
(b) fluxes are maximized on the outgoing roads or at least a regulation of those fluxes is possible.

8.1.2 Analysis of Backward Shock Waves on the Incoming Roads

The next proposition establishes a necessary and sufficient condition on the X_i's and η for (a) to hold.

Proposition 8.1.1. *With the notations above, backward shock waves created in the incoming roads behave in the following way:*

$(S1)$ *The backward shock waves created in the road I_1 remains bounded if and only if $X_1 > 0$ and $1 - X_1^2 \leq \eta$;*
$(S2)$ *The backward shock waves created in the road I_2 remains bounded if and only if $X_2 > 0$ and $\eta \leq X_2^2$.*

For a proof of the previous proposition, the reader may refer to [26].

Combining $(S1)$ and $(S2)$, one gets that Issue (a) has a positive solution if and only if

$$X_i > 0, \quad i = 1, 2 \text{ and } X_1^2 + X_2^2 \geq 1, \tag{8.1.3}$$

and

$$1 - X_1^2 \leq \eta \leq X_2^2. \tag{8.1.4}$$

Rewriting (8.1.3) in terms of the ρ_i leads to

$$\bar{\rho}_i < \frac{1}{2} \quad i = 1, 2 \quad \text{and} \quad \left(\frac{1}{2} - \bar{\rho}_1\right)^2 + \left(\frac{1}{2} - \bar{\rho}_2\right)^2 \geq \frac{1}{4}.$$

Assuming for instance that $\bar{\rho}_1 = \bar{\rho}_2 = \bar{\rho}$, the previous condition reduces to $\bar{\rho} \leq \frac{\sqrt{2}-1}{2\sqrt{2}}$. Therefore, at the light of (8.1.3), one can see that (a) holds only for low traffic.

8.1.3 Asymptotic Regime for the Outgoing Roads

We investigate the asymptotic regime on each of the two outgoing roads in order to address issue (b).

The main result of this subsection gives the value of the asymptotic flux on each of the two outgoing roads in terms of data of the incoming roads and of the distribution coefficients. By asymptotic flux, we mean the limit, as t, x go both to infinity, of the flux function $f(\rho)$. It is clear that, for each outgoing road, the flux function at the junction is periodic (of period $\Delta_r + \Delta_g$) and piecewise constant. More precisely, there exist four non-overlapping time intervals L_1, L_2, L_3, L_4 (L_1 and L_2 have strictly positive length, while L_3 and L_4 could be empty) so that, if we denote with $T^i \geqslant 0$ the length of each L_i, then $\Delta_r + \Delta_g = T^1 + T^2 + T^3 + T^4$ and the flux on on L_i is constantly equal to f^i. Then, we have the following result: the asymptotic flux on each of the two outgoing roads is equal to the average of the entering flux, at the junction, over a time period.

The above statement is just a particular case of Proposition 8.1.3, given below, which is of independent interest. Before stating it, we need the next definition.

Definition 8.1.2. *Let n be a positive integer. We say that $B : [0, \infty) \to [0, 1]$ is an admissible boundary datum with n jumps if B has the following properties. There exist $T > 0$, $T_0 > 0, \cdots, T_n > 0$ with $T_0 + \cdots + T_n = T$, and $n + 1$ real numbers $\rho_0, \cdots, \rho_n \in [0, 1]$ such that*

1. *$\rho_i \neq \rho_{i+1}$ for every $i \in \{0, \ldots, n-1\}$;*
2. *$\rho_0 = \max\{\rho_0, \ldots, \rho_n\}$;*
3. *$\rho_n = \min\{\rho_0, \ldots, \rho_n\}$;*
4. *the function B is defined as*

$$B(t) = \begin{cases} \rho_0, & \text{if } t \in \cup_{k \in \mathbb{N}} [kT, kT + T_0), \\ \rho_1, & \text{if } t \in \cup_{k \in \mathbb{N}} [kT + T_0, kT + T_0 + T_1), \\ \vdots & \vdots \\ \rho_n, & \text{if } t \in \cup_{k \in \mathbb{N}} [kT + T_0 + \cdots + T_{n-1}, (k+1)T). \end{cases} \tag{8.1.5}$$

Proposition 8.1.3. *Let B be an admissible boundary datum with n jumps, $n \geq 1$. Consider the solution $\rho(t, x)$ on an outgoing road associated to B. Then, as t and x tend both to infinity, ρ tends to some $\rho^{asymp} \in [0, \sigma]$ defined by*

$$f(\rho^{asymp}) = \frac{1}{T} \sum_{i=0}^{n} T_i f(\rho_i), \tag{8.1.6}$$

where T_i and ρ_i are given by (8.1.5).

For a proof see [26].

For the outgoing roads I_3 and I_4, the admissible boundary data admit, in general, three jumps. We can explicit them, by taking into account equations

(8.1.1) and (8.1.2). We use $f^{-1} : [0, f(\sigma)] \to [0, \sigma]$ to denote the inverse function of f on the ad hoc intervals.

In the next corollary, we apply Proposition 8.1.3 and obtain the values of f_3^{asymp} and f_4^{asymp}, the asymptotic fluxes on road I_3 and road I_4. For simplicity of the results, we assume that $X_1, X_2 > 0$. Moreover, we define

$$\bar{f}_3 := \alpha f(\bar{\rho}_1) + \beta f(\bar{\rho}_2), \quad \bar{f}_4 := (1-\alpha) f(\bar{\rho}_1) + (1-\beta) f(\bar{\rho}_2), \quad F := f(\bar{\rho}_1) + f(\bar{\rho}_2).$$

Corollary 8.1.4. *Assume that $X_1, X_2 > 0$. Then, the following holds:*

(*i*) *if $\eta < \min\{X_2^2, 1 - X_1^2\}$, then road I_1 gets stuck, road I_2 remains free and*

$$f_3^{asymp} = \bar{f}_3 - \alpha f(\sigma)[(1 - X_1^2) - \eta], \tag{8.1.7}$$

$$f_4^{asymp} = \bar{f}_4 - (1 - \alpha) f(\sigma)[(1 - X_1^2) - \eta]. \tag{8.1.8}$$

The total flux going through the junction over a time period of $\Delta_r + \Delta_g$ is equal to $F - f(\sigma)[(1 - X_1^2) - \eta]$;

(*ii*) *if $\eta > \max\{X_2^2, 1 - X_1^2\}$, then road I_2 gets stuck, road I_1 remains free and*

$$f_3^{asymp} = \bar{f}_3 - \beta f(\sigma)(\eta - X_2^2), \tag{8.1.9}$$

$$f_4^{asymp} = \bar{f}_4 - (1 - \beta) f(\sigma)(\eta - X_2^2). \tag{8.1.10}$$

The total flux going through the junction over a time period of $\Delta_r + \Delta_g$ is equal to $F - f(\sigma)(\eta - X_2^2)$;

(*iii*) *If $\eta \in [1 - X_1^2, X_2^2]$, then both roads I_1 and I_2 remain free and*

$$f_3^{asymp} = \bar{f}_3, \quad f_4^{asymp} = \bar{f}_4; \tag{8.1.11}$$

The total flux going through the junction over a time period of $\Delta_r + \Delta_g$ is equal to F;

(*iv*) *If $\eta \in (X_2^2, 1 - X_1^2)$, then both roads I_1 and I_2 get stuck and*

$$f_3^{asymp} = f(\sigma)[(1 - \eta)\beta + \eta\alpha], \quad f_4^{asymp} = f(\sigma)[(1 - \eta)(1 - \beta) + \eta(1 - \alpha)]. \tag{8.1.12}$$

The total flux going through the junction over a time period of $\Delta_r + \Delta_g$ is equal to $2f(\sigma)$.

8.2 Single Lane Traffic Circle with low Traffic

In this section we fix a simple network representing a traffic circle and assume that there is low traffic, in the sense that the number of cars reaching the circle is less then the capacity of the circle itself.

There are four roads, named I_1, I_2, I_3, I_4, the first two incoming in the circle and the other two outgoing. Moreover there are four roads I_{1R}, I_{2R}, I_{3R}, I_{4R} that form the circle as in Figure 8.2.

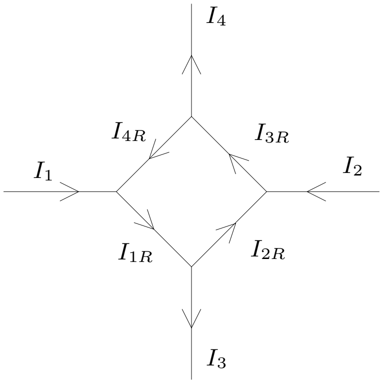


Fig. 8.2. Traffic circle.

As above, roads are parameterized by $[a_i, b_i]$, $i = 1, \ldots, 4$, and $[a_{iR}, b_{iR}]$, $i = 1, \ldots, 4$. It is natural to assign a traffic distribution matrix A to describe how traffic coming from roads I_1, I_2 choose to exit to roads I_3 and I_4. Indeed the roads of the circle are just intermediate towards the final destination. Thus we assume to have two fixed parameters: $\alpha, \beta \in]0, 1[$ so that:

(C1) If M cars reach the circle from road I_1, then αM drive to road I_3 and $(1-\alpha)M$ drive to road I_4;
(C2) If M cars reach the circle from road I_2, then βM drive to road I_4 and $(1-\beta)M$ drive to road I_3.

Let us first consider a static situation. Let $\bar{\rho}_1$ and $\bar{\rho}_2$ be constant densities from the roads I_1 and I_2 respectively (this amounts to fix boundary conditions):

$$\rho_1(t, a_1) \equiv \bar{\rho}_1, \qquad \rho_2(t, a_2) \equiv \bar{\rho}_2. \tag{8.2.13}$$

If the roads I_3 and I_4 can absorb all incoming traffic, e.g. if

$$f(\bar{\rho}_1) + f(\bar{\rho}_2) \le f(\sigma), \tag{8.2.14}$$

then we should reach the situation of Figure 8.3, where the constant fluxes on each road are written.

For this to happen we must have that the coefficients for the crossing (I_{1R}, I_3, I_{2R}) are

$$\alpha_{1R,3} = \frac{\alpha f(\bar{\rho}_1) + (1-\beta) f(\bar{\rho}_2)}{f(\bar{\rho}_1) + (1-\beta) f(\bar{\rho}_2)}, \quad \alpha_{1R,2R} = \frac{(1-\alpha) f(\bar{\rho}_1)}{f(\bar{\rho}_1) + (1-\beta) f(\bar{\rho}_2)}, \tag{8.2.15}$$

and similarly for (I_{3R}, I_4, I_{4R})

$$\alpha_{3R,4} = \frac{(1-\alpha) f(\bar{\rho}_1) + \beta f(\bar{\rho}_2)}{(1-\alpha) f(\bar{\rho}_1) + f(\bar{\rho}_2)}, \quad \alpha_{3R,4R} = \frac{(1-\beta) f(\bar{\rho}_2)}{(1-\alpha) f(\bar{\rho}_1) + f(\bar{\rho}_2)}. \tag{8.2.16}$$

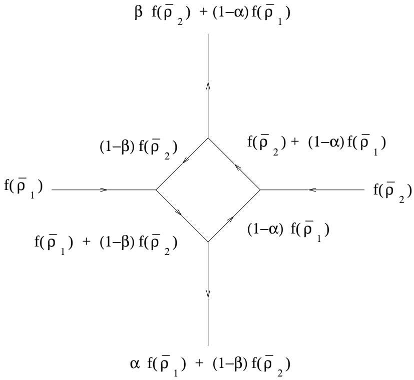


Fig. 8.3. Equilibrium for traffic circle

If the network is initially empty, the boundary data are given by (8.2.13) and (8.2.14) holds true, then firstly the cars from road I_1 and I_2 reach road I_3 and I_4 respectively and the coefficients should be simply set as:

$$\alpha_{1R,3} = \alpha, \qquad \alpha_{1R,2R} = (1-\alpha), \qquad \alpha_{3R,4} = \beta, \qquad \alpha_{3R,4R} = (1-\beta). \tag{8.2.17}$$

However, then also cars from road I_2 reach road I_3 (and cars from road I_1 reach road I_3) then we should modify in time the coefficients and finally set as in (8.2.15) and (8.2.16). With this choice, there exists $T > 0$ such that the solution is given by the fluxes indicated in Figure 8.3 for every $t \geq T$. Thus we see that the model is working properly at not too heavy traffic level, i.e. for which (8.2.14) holds true. However, it is necessary to let the coefficients α vary on time, more precisely:

Theorem 8.2.1. *Consider the circle network and assume (8.2.13), (8.2.14). There exists time dependent coefficients* $\alpha : [0, +\infty) \to [0, 1]$*, with (8.2.17) holding at time* 0 *and (8.2.15), (8.2.16) for large enough times, and* $T > 0$ *such that the solution* $\rho(t)$ *is constantly equal to that of Figure 8.3 for every* $t \geq T$.

Proof. We construct solution by wave front tracking. Cars from roads I_1 and I_2 reach outgoing roads I_3 and I_4 via rarefaction fans. At each time a rarefaction shock reach either the junction (I_{1R}, I_{2R}, I_4) or the junction (I_{3R}, I_{4R}, I_3) we adjust the corresponding coefficients in such a way that cars reach the correct road. □

8.3 Single Lane Traffic Circle with heavy Traffic

We consider the case of a circle as in the previous section, but for which the condition (8.2.14) is violated. More precisely we have possible traffic jams if one of the following conditions holds:

$$f(\bar\rho_1) + (1-\beta)f(\bar\rho_2) > f(\sigma), \tag{8.3.18}$$

$$(1-\alpha)f(\bar\rho_1) + f(\bar\rho_2) > f(\sigma). \tag{8.3.19}$$

Assume that we are in situation of the traffic equilibrium for low traffic (Figure 8.3) but now with conditions (8.3.18), (8.3.19) holding true. Then immediately shocks are produced on some roads of the junctions (I_1, I_{4R}, I_{1R}) and (I_2, I_{2R}, I_{3R}).

Remark 8.3.1. Notice that if we start from empty circle then rarefaction waves start to fill up the circle reaching at some point a situation as in Figure 8.3.

From now on, for simplicity, we adopt the following notation.

Notation. Each wave will be indicated as (f_l, f_r) where f_l is the value of the flux to the left of the wave and f_r the value of the flux to the right, being clear from the context which are the values of ρ on the left and right of the wave.

For simplicity we set:

$$f_1 := f(\bar\rho_1), \qquad f_2 := f(\bar\rho_2),$$

$$q_1 := q_{(I_1, I_{4R}, I_{1R})}, \qquad q_2 := q_{(I_2, I_{2R}, I_{3R})},$$

where the q's denote the right of way parameters.

First consider the junction (I_1, I_{4R}, I_{1R}); we have:

$$\gamma_1^{max} = f_1,\ \gamma_{4R}^{max} = (1-\beta)f_2\ ,\gamma_{1R}^{max} = f(\sigma) = 1.$$

Then we have $\hat\gamma_{1R} = f(\sigma) = 1$, hence $\hat\rho_{1R} = \sigma$, and we have to distinguish three cases, depending on the value of q_1:

a) $q_1 \le 1 - \frac{(1-\beta)f_2}{f(\sigma)}$, then $\hat\gamma_1 = f(\sigma) - (1-\beta)f_2$ and $\hat\gamma_{4R} = (1-\beta)f_2$;
b) $1 - \frac{(1-\beta)f_2}{f(\sigma)} < q_1 < \frac{f_1}{f(\sigma)}$, then $\hat\gamma_1 = q\, f(\sigma)$ and $\hat\gamma_{4R} = (1-q)\, f(\sigma)$;
c) $q_1 \ge \frac{f_1}{f(\sigma)}$, then $\hat\gamma_1 = f_1$ and $\hat\gamma_{4R} = f(\sigma) - f_1$.

In case a) a shock is produced on road I_1 and no wave on road I_{4R}, in case c) a a shock is produced on road I_{4R} and no wave on road I_1, finally in case b) a shock is produced on both roads.

The analysis of junction (I_2, I_{2R}, I_{3R}) is absolutely similar.

Assume first that we are in case a) for both junctions (I_1, I_{4R}, I_{1R}) and (I_2, I_{2R}, I_{3R}). Then rarefactions are produced on roads I_{1R} and I_{3R}.

Notice that the flux $f(\sigma)$ on road I_{2R} is composed of $f(\sigma)-(1-\beta)f_2$ from road I_1 and $(1-\beta)f_2$ from road I_{4R}. Since α of flux from road I_1 exits to road I_3 and all flux from I_{4R} exits to road I_3, we have that $\alpha f(\sigma)+(1-\alpha)(1-\beta)f_2$ exits to road I_3 and $(1-\alpha)(f(\sigma)-(1-\beta)f_2)$ proceeds to I_{2R}. From condition

(8.3.18), it follows that $(1-\alpha)f_1 > (1-\alpha)(f(\sigma)-(1-\beta)f_2)$, thus on road I_{2R} it generates a rarefaction that reaches the crossing (I_2, I_{2R}, I_{3R}). This rarefaction decreases the traffic flux entering from road I_{2R} thus the circle is not stuck. The analysis is the same for roads I_{3R} and I_{4R}.

Finally we obtain the following

Proposition 8.3.2. *The traffic on the circle never gets stuck if the following holds:*

$$q_1 \leq 1 - \frac{(1-\beta)f_2}{f(\sigma)}, \qquad q_2 \leq 1 - \frac{(1-\alpha)f_1}{f(\sigma)}.$$

Assume now that we are in case b) for both junctions (I_1, I_{4R}, I_{1R}) and (I_2, I_{2R}, I_{3R}). Then rarefactions are produced on roads I_{1R} and I_{3R} and shocks on the other roads.

Notice that the flux $f(\sigma)$ on road I_{2R} is composed of $q_1 f(\sigma)$ from road I_1 and $(1-q_1)f(\sigma)$ from road I_{4R}. Since α of flux from road I_1 exits to road I_3 and all flux from I_{4R} exits to road I_3, we have that $(1-(1-\alpha)q_1)f(\sigma)$ exits to road I_3 and $(1-\alpha)q_1 f(\sigma)$ proceeds to I_{2R}. This is the same as saying that:

$$\alpha_{1R,2R} = (1-\alpha)q_1.$$

Then when the shock produced on road I_{2R} reaches the junction (I_2, I_{2R}, I_{3R}), then a shock is produces on I_{1R}:

$$\left(f(\sigma), \frac{(1-q_2)f(\sigma)}{(1-\alpha)q_1}\right).$$

We can perform the same analysis on roads I_{3R} and I_{4R}. Then shocks are produced on the whole circle recursively.

To describe the evolution we introduce some more notation:

- x_1^n is the value of the flux on road I_{1R} after the n-th shock-junction interaction.
- x_2^n is the value of the flux on road I_{2R} after the n-th shock-junction interaction.
- x_3^n is the value of the flux on road I_{3R} after the n-th shock-junction interaction.
- x_4^n is the value of the flux on road I_{4R} after the n-th shock-junction interaction.
- x_5^n is the value of the flux on road I_1 after the n-th shock-junction interaction.
- x_6^n is the value of the flux on road I_2 after the n-th shock-junction interaction.

Defining $x^n \in \mathbb{R}^6$ to be the vector with x_i^n as components, we get the following evolution:

$$x^{n+1} = A x^n \tag{8.3.20}$$

where

$$A := \begin{pmatrix} 0 & 0 & \frac{1-q_2}{(1-\alpha)q_1} & 0 & 0 & 0 \\ 0 & 0 & 1-q_2 & 0 & 0 & 0 \\ \frac{1-q_1}{(1-\beta)q_2} & 0 & 0 & 0 & 0 & 0 \\ (1-q_1) & 0 & 0 & 0 & 0 & 0 \\ q_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & q_2 & 0 & 0 & 0 \end{pmatrix}.$$

So one easily see that it is enough to consider the evolution of variables x_1^n and x_3^n. The corresponding reduced matrix is

$$\widetilde{A} := \begin{pmatrix} \frac{1-q_2}{(1-\alpha)q_1} & 0 \\ 0 & \frac{1-q_1}{(1-\beta)q_2} \end{pmatrix}.$$

The eigenvalues are given by the equation:

$$\lambda^2 = \frac{(1-q_1)(1-q_2)}{(1-\alpha)(1-\beta)q_1 q_2}.$$

Thus we get the following:

Proposition 8.3.3. *Assume that:*

$$1 - \frac{(1-\beta)f_2}{f(\sigma)} < q_1 < \frac{f_1}{f(\sigma)},$$

$$1 - \frac{(1-\alpha)f_1}{f(\sigma)} < q_2 < \frac{f_2}{f(\sigma)}.$$

Then the traffic flow does not stop if the following holds:

$$\frac{(1-q_1)(1-q_2)}{(1-\alpha)(1-\beta)q_1 q_2} > 1. \tag{8.3.21}$$

If, finally, case c) (strictly) hold for both junctions (I_1, I_{4R}, I_{1R}) and (I_2, I_{2R}, I_{3R}), then rarefactions are produced on roads I_{1R} and I_{3R} and shocks on roads I_{2R} and I_{4R}. Moreover we get the inequalities:

$$q_1 f(\sigma) < f_1, \qquad q_2 f(\sigma) < f_2,$$

$$(1-q_1)f(\sigma) < (1-\alpha)f_1, \qquad (1-q_2)f(\sigma) < (1-\beta)f_2,$$

and one easily checks that condition (8.3.21) can not hold. Therefore we get the following:

Proposition 8.3.4. *Assume that:*

$$q_1 > \frac{f_1}{f(\sigma)},$$

$$q_2 > \frac{f_2}{f(\sigma)},$$

then the circle does get stuck.

8.4 Multi Lane Traffic Circle with no Interaction

We propose a second model for the traffic circle. We assume that:

(ML) The traffic rounding in the circle and that entering the circle flow on independent lanes.

This is the case if the entering lanes have sufficiently long ramps. Then we can model the traffic circle assuming that lanes are distinct roads that cross when there is an exiting lane, thus we are in the situation depicted in Figure 8.4.

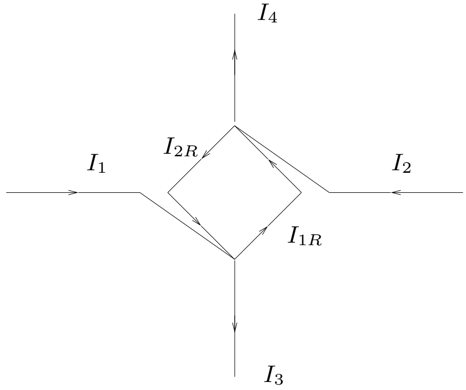


Fig. 8.4. Traffic circle with multi lane not interacting.

To determine completely the model we have to provide the traffic distribution matrix for the junctions $((I_1, I_{2R}), (I_3, I_{1R}))$ and $((I_2, I_{1R}), (I_4, I_{2R}))$. Regarding the first crossing we have that the traffic from road I_1 should distribute to (I_3, I_{1R}) according to the coefficients $(\alpha, 1-\alpha)$, while the traffic from road I_{2R} should not enter road I_{1R}. Finally we get the coefficients:

$$\alpha_{1,3} = \alpha, \qquad \alpha_{1,1R} = 1 - \alpha, \qquad \alpha_{2R,3} = 1, \qquad \alpha_{2R,1R} = 0. \tag{8.4.22}$$

Similarly for the other junction we set:

$$\alpha_{2,4} = \beta, \qquad \alpha_{2,2R} = 1 - \beta, \qquad \alpha_{1R,4} = 1, \qquad \alpha_{1R,2R} = 0. \tag{8.4.23}$$

Let us determine the solution to a Riemann problem at a junction $((I_1, I_{2R}), (I_3, I_{1R}))$. The region of admissible fluxes Ω is determined by the constraints

$$\gamma_1 \le \gamma_1^{max}, \qquad \gamma_{2R} \le \gamma_{2R}^{max},$$

$$\alpha\gamma_1 + \gamma_{2R} \le \gamma_3^{max}, \qquad (1-\alpha)\gamma_1 \le \gamma_{1R}^{max}.$$

Since $\alpha < 1$ we get that the solution is always obtained for:

$$\hat{\gamma}_1 = \min\{\gamma_1^{max}, \frac{\gamma_{1R}^{max}}{1-\alpha}\}.$$

Therefore all possible traffic from road I_1 flow through the circle while the traffic from road I_{2R} may be subject to restriction. This rule models appropriately the situation in which cars in traffic circle must yield.

We thus obtain the following:

Proposition 8.4.1. *The model of Figure 8.4 corresponds to the situation of cars in the traffic circle yielding to entering traffic.*

Direct calculations show the following:

Proposition 8.4.2. *If condition (8.2.14) is verified then the solution reach the equilibrium showed in Figure 8.5.*

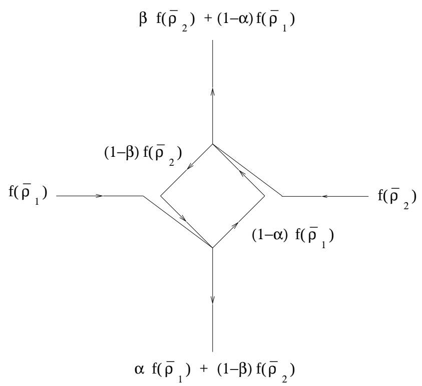


Fig. 8.5. Equilibrium for traffic circle with multi lane not interacting.

8.5 Traffic Light vs Traffic Circle

We discuss the results obtained in previous sections to compare the possibility of traffic control offered by a junction with a traffic light and one with a traffic circle. We start analyzing the situation of low traffic and then that of heavy traffic.

8.5.1 Low Traffic

Low traffic for a traffic light means that conditions (8.1.3) are verified. Now if the choice of η according to (8.1.4) is possible, then the traffic flow on outgoing roads equal that on incoming ones $f(\bar{\rho}_1) + f(\bar{\rho}_2)$. This corresponds to the fact that the traffic on incoming roads is not blocked. On the other side, if $X_2^2 < 1 - X_1^2$ then some regulation of traffic is possible again by tuning η. For example, since not both incoming roads can avoid a stuck situation we can choose one to be free of jams. In the same way also some regulation of outgoing traffic is possible.

For the single lane traffic circle, low traffic corresponds to condition (8.2.14), or more precisely to (8.3.18),(8.3.19) being false. In this case there is no jam on incoming roads and the outgoing flux equals the incoming ones. However, there is no possibility of flux regulation on outgoing roads.

The situation of multi lane traffic circle is completely similar to the single lane case.

The results are summarized in Figure 8.6.

Low traffic	Traffic light	Single lane traffic circle	Multi lane traffic circle
Traffic on incoming roads	Possibly stuck	Not stuck	Not stuck
Outgoing flux	Maximized under some conditions	Maximized	Maximized
Traffic regulation	Yes by choosing η	No	No

Fig. 8.6. Light vs circle, low traffic case.

8.5.2 Heavy Traffic

In case of heavy traffic, the traffic light surely provokes jams on incoming roads with the effect of lowering the outgoing flux. This happens if conditions (8.1.3),(8.1.4) are violated. However, there is always a minimum amount of traffic going through that can be regulated by the parameter η.

The single lane traffic circle performs in good way if the traffic circle is not completely stopped. Thanks to the discussion of Section 8.3, in particular Propositions 8.3.2 and 8.3.3, a stuck traffic may happen only for quite high traffic levels and can be avoided in most cases regulating the right of way parameters.

The multi lane traffic circle is not performing well for heavy traffic. Indeed, thanks to Propositions 8.4.1, we see that the probability of provoking a jam is very high.

The results are summarized in Figure 8.7.

Heavy traffic	Traffic light	Single lane traffic circle	Multi lane traffic circle
Traffic on incoming roads	Stuck	Stuck	Stuck
Outgoing flux	Always positive	Maximized choosing right of way parameters	Probably stuck
Traffic regulation	Yes	No	No

Fig. 8.7. Light vs circle, heavy traffic case.

8.5.3 Comparison

The performance of traffic light and circle is quite different also depending on the entity of traffic: low or high. In the former case the traffic circle (both single and multi lane) is preferable for maximizing the outgoing flow. The traffic light could be convenient only in the case of big difference between road sizes when a regulation is in order.

In case of heavy traffic, traffic light is surely guaranteeing a minimum level of outgoing flux, but the maximum level can be quite bounded. The single lane traffic circle performs in a satisfactory way if the circle itself is not stuck. This can be avoided by regulating the right of way parameters. Finally multi lane traffic circle presents a high probability of traffic jams.

Concluding, specific situations (such as expected low traffic or big difference among roads) can render one solution preferable to the other as illustrated above. For a general purpose junction, a single lane traffic circle seems the best solution with an appropriate choice of the right of way rules. The possibility of putting traffic lights at crossings of the circle, working only in heavy traffic situation, could lower further the probability of creating jams.

9

Telecommunication Networks (by C. D'Apice and R. Manzo)

In this chapter we propose a macroscopic fluid dynamic model dealing with the flows of information on a telecommunication network encoded in packets. The analogy with fluids comes from considering packets as particles. However, we have to take into account packet loss effects (which were not considered in car traffic modelling.) Our idea is to look at the network at an intermediate time scale so that packets transmission happens at a faster level but the equilibria of the whole network are reached only as asymptotic. This permits to construct a model relying on macroscopic description.

9.1 Introduction

There exist various approaches to traffic flow on telecommunication networks, in particular for Internet and with special focus on properties of control congestion algorithms as TCP/IP, see for example [13, 70, 108]. In these approaches the network is essentially *blind* and the control on possible congestions is exerted by the higher level agents as TCP and other protocols. Our idea is rather to take a large number of nodes, which use some simple routing algorithm, and via some limiting procedure obtain a partial differential equation for the packet density on the network. First we focus on a straight transmission line and justify formally the limiting procedure. Then we pass to consider a network and introduce a routing algorithm for nodes with many entering and exiting lines.

A telecommunication network is formed by a finite collection of transmission lines and nodes. We assume that each node receives and sends information encoded in packets. Having in mind Internet as key model, it is assumed that:

Rule 1) Each packet travels on the network with a fixed speed and with assigned final destination.

Rule 2) Nodes receive, process and then forward packets. Packets may be lost with a probability increasing with the number of packets to be processed. Each lost packet is sent again.

We first model the behavior of a single straight transmission line on which there are some consecutive nodes. Each node sends packets to the following one a first time, then packets which are lost in this process are sent a second time and so on. The important point is that each packet is sent until it reaches next node, thus, looking at macroscopic level, it is assumed that packets are conserved. This leads for the microscopic dynamics to the simple model consisting of a single conservation law:

$$\rho_t + f(\rho)_x = 0, \tag{9.1.1}$$

where ρ is the packet density, v is the velocity and $f(\rho) = v\rho$ is the flux. Since the packet transmission velocity on the line is assumed constant, we can derive an average transmission velocity among nodes considering the amount of packets that may be lost. More precisely, assigning a loss probability as function of the density, it is possible to compute a velocity function and thus a flux function.
The conclusion is rigorously justified only for constant density, but is assumed to hold in general. This corresponds to the hypotheses that macroscopic density waves move at a velocity much smaller than the packets transmission velocity. Even if our limiting procedure is not completely rigorous, there are other similar approaches, as [8] for supply chains, which lead to conservation laws using some weak convergence. Moreover, since our method to solve problems at nodes is based only on flux values, every limiting procedure, which leads to a conservation law formulation, may be used to treat the problem on a network.

Then we introduce a way of solving dynamics at nodes in which many lines intersect. For this, respecting Rule 2), we propose a routing algorithm:

(RA) Packets are sent to outgoing lines in order to maximize the flux though the node.

The main differences of (RA) with the algorithm of Chapter 5 are the following. The latter simply sends each packet to the outgoing line which is naturally chosen according to the final destination of the packet itself. The algorithm is blind to possible overloads of some outgoing lines and, by some abuse of notation, is similar to the behavior of a "switch". The former algorithm, on the contrary, send packets to outgoing lines taking into account the loads, and thus possibly redirecting packets. Again by some abuse of notation, this is similar to a "router" behavior.

Notice that this second algorithm was not considered for car traffic, because redirection of cars is not often expected from modelling point of view (except special situations as closure of a road).

In order to determine unique solutions to Riemann problems, some additional parameters are introduced, called respectively priority parameters and traffic distribution parameters. The former describe priorities among incoming lines, while the latter have the same meaning of the traffic distribution matrix.

The advantage of this algorithm is that the flux variation at a node is conserved for interaction of waves from transmission lines. This permits us both to obtain estimates on the total variation of density, thus to construct solutions again by wave-font tracking, and also to obtain uniqueness and Lipschitz continuous dependence of solutions. The latter result is achieved by the method introduced in [18, 20] and illustrated in Section 2.7, which considers a Riemannian type metric on L^1. More precisely, the distance among solutions is measured by paths in L^1 which admit some generalized tangent vectors. The key point is that the norms of tangent vectors are known to decrease inside each line (i.e. for scalar conservation laws), while for interactions with nodes its evolution is determined by flux variation. As explained below in Section 9.5, other known methods, to treat uniqueness for scalar conservation laws, seem not to work for the network case.

The obtained results show the strong effect of the routing algorithm. More precisely, **the choice of a "router" type algorithm, i.e. (RA), implies stability of solutions, with respect of perturbation of the data, opposed to the instability obtained with the "switch" type ones.**

9.2 Packets Loss and Velocity Functions on Transmission Lines

Consider a transmission line formed by a sequence of nodes N_k, representing routers, and edges which connect consecutive nodes. Thus the transmission line is represented by a real interval I union of many edges and nodes.

Each node (router) sends and receives packets. Following Rule 1), we assume that packets flow at constant velocity from each node N_k to N_{k+1}. Taking a discrete time scale for the evolution, the state at time t_i is described by the packets quantities $R_k(t_i)$ on nodes N_k and transmission happens among consecutive nodes between two discrete times. Therefore, to determine the dynamics on I we need to describe the effect of packets loss on the velocity of transmission function.

We assume that each node N_k sends again packets that are lost by the following node N_{k+1}. Therefore the number of packets is conserved, i.e. at macroscopic level we expect (9.1.1) to hold. More precisely, we assume that there exists a function $p : [0, R_{max}] \to [0, 1]$ which assigns the packet loss probability as function of the number of packets.

Let us focus now on two consecutive nodes and introduce some notation. Suppose that δ is the distance between the nodes N_k and N_{k+1}. Let Δt_0 be the transmission time of packets from node N_k to node N_{k+1} if they are sent with success at the first attempt, and Δt_{av} the average transmission time when some packets are lost by N_{k+1}. Finally, we denote with $\bar{v} = \frac{\delta}{\Delta t_0}$ and $v = \frac{\delta}{\Delta t_{av}}$ the packets velocity in the two cases.

At the first attempt, the packets sent by node N_k reach with success node N_{k+1} with probability $(1-p)$ and they are lost by node N_{k+1} with probability p. At the second attempt there are p of the total number of packets left to be sent again and $(1-p)p$ are sent with success while p^2 are lost. Going on at the n-th attempt $(1-p)p^{n-1}$ packets are sent successfully and p^n are lost. The average transmission time is equal to

$$\Delta t_{av} = \sum_{n=1}^{+\infty} n \Delta t_0 (1-p) p^{n-1} = \frac{\Delta t_0}{1-p}, \tag{9.2.2}$$

from which we get that the transmission velocity is given by

$$v = \frac{\delta}{\Delta t_{av}} = \frac{\delta}{\Delta t_0}(1-p) = \bar{v}(1-p). \tag{9.2.3}$$

The above reasoning works for the entire line if $R_k(t_0) = R$ for all k. In fact, one gets immediately that $R_k(t_i) = R$ for all i and k thus it holds:

Lemma 9.2.1. *Assume that $R_k(t_0) = R$ for all k. Then the average transmission time and velocity are given by (9.2.2) and (9.2.3).*

Clearly Lemma 9.2.1 gives an average velocity only if the density is constant. However, we assume the conclusion to hold in general for the macroscopic velocity and use this together with equation (9.1.1). This assumptions is not completely justified but it is reasonable if the transmission velocity of packets is expected to be much bigger than the macroscopic velocity.

We may also assign the loss probability directly as function of the packet density, then the corresponding flux is easily determined. Such loss probability should vanish for low load levels of nodes and reach the value 1 for $R = R_{max}$. We show some choice of packets loss functions and the corresponding macroscopic fluxes.

Example 9.2.2. Let us suppose that the packets loss probability is given by

$$p(\rho) = \begin{cases} 0, & 0 \le \rho \le \sigma, \\ \frac{2(\rho-\sigma)}{\rho}, & \sigma \le \rho \le \rho_{\max}, \end{cases}$$

for some $\sigma \in]0, \rho_{max}[$; see Figure 9.1.

Then the average transmission velocity is equal to

$$v(\rho) = \bar{v}(1 - p(\rho)) = \begin{cases} \bar{v}, & 0 \le \rho \le \sigma, \\ \bar{v}\frac{(2\sigma-\rho)}{\rho}, & \sigma \le \rho \le \rho_{\max}. \end{cases}$$

Imposing that

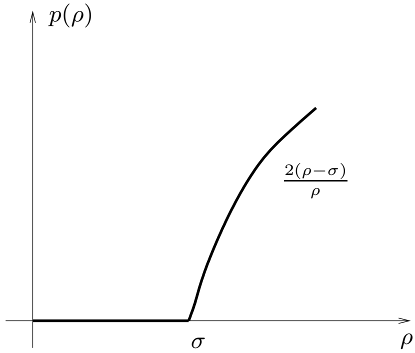


Fig. 9.1. Packets loss function.

$$v(\rho_{\max}) = \bar{v}\frac{(2\sigma - \rho_{\max})}{\rho_{\max}} = 0,$$

we get that $\sigma = \frac{\rho_{\max}}{2}$. Since $f(\rho) = v(\rho)\rho$ it follows that

$$f(\rho) = \begin{cases} \bar{v}\rho, & 0 \le \rho \le \sigma, \\ \bar{v}(2\sigma - \rho), & \sigma \le \rho \le \rho_{\max}. \end{cases}$$

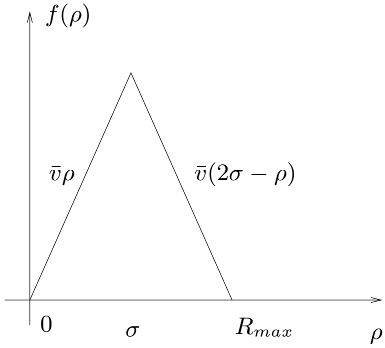


Fig. 9.2. Flux function.

Example 9.2.3. Suppose that

$$p(\rho) = \begin{cases} 0, & 0 \le \rho \le \sigma, \\ \frac{\rho-\sigma}{\sigma}, & \sigma \le \rho \le \rho_{\max}. \end{cases}$$

It follows that

$$v(\rho) = \begin{cases} \bar{v}, & 0 \le \rho \le \sigma, \\ \frac{\bar{v}(2\sigma-\rho)}{\sigma}, & \sigma \le \rho \le \rho_{\max}, \end{cases}$$

and

$$f(\rho) = \begin{cases} \bar{v}\rho, & 0 \le \rho \le \sigma, \\ \frac{\bar{v}\rho(2\sigma-\rho)}{\sigma}, & \sigma \le \rho \le \rho_{\max}. \end{cases}$$

Example 9.2.4. Suppose that

$$p(\rho) = \begin{cases} 0, & 0 \le \rho \le \sigma, \\ \frac{(\rho-\sigma)^2}{\sigma^2}, & \sigma \le \rho \le \rho_{\max}. \end{cases}$$

It follows that

$$v(\rho) = \begin{cases} \bar{v}, & 0 \le \rho \le \sigma, \\ \frac{\bar{v}\rho(2\sigma-\rho)}{\sigma^2}, & \sigma \le \rho \le \rho_{\max}, \end{cases}$$

and

$$f(\rho) = \begin{cases} \bar{v}\rho, & 0 \le \rho \le \sigma, \\ \frac{\bar{v}\rho^2(2\sigma-\rho)}{\sigma^2}, & \sigma \le \rho \le \rho_{\max}. \end{cases}$$

Remark 9.2.5. Examples 9.2.2 and 9.2.3 lead to fluxes which are not $\mathcal{C}^1$, the opposite happens for Example 9.2.4 Notice that only for Example 9.2.2 the corresponding flux has the property that, for every $\tilde{\rho} \in]0, \rho_{max}[$,

$$\lim_{\rho \to \tilde{\rho}^+} f'(\rho) \neq 0, \qquad \text{and} \qquad \lim_{\rho \to \tilde{\rho}^-} f'(\rho) \neq 0.$$

Thus the density variation along discontinuities not crossing σ is equivalent to the flux ones.

In what follows we suppose that measures on packets loss probability lead to the formulation of Example 9.2.2. This allows to control the variation of the density function in terms of the variation of the flux function.

We can suppose for simplicity that $\rho_{\max} = 1$, so we have the following assumptions on the flux:

(F) $f : [0,1] \to R, \quad f(\rho) = \begin{cases} \bar{v}\rho, & 0 \le \rho \le \sigma, \\ \bar{v}(2\sigma - \rho), & \sigma \le \rho \le 1, \end{cases}$
$f(0) = f(1) = 0$. Thus $\sigma = \frac{1}{2}$ is the unique maximum point.

9.3 Riemann Solver at Nodes

In this section we describe the Riemann solver at nodes following the routing algorithm (RA).

To solve Riemann problems according to (RA) we need some additional parameters called priority and traffic distribution parameters. For simplicity of exposition, consider first a junction J in which there are two transmission lines with incoming traffic and two transmission lines with outgoing traffic. In this case we have only one priority parameter $q \in]0,1[$ and one traffic distribution parameter $\alpha \in]0,1[$. We denote with $\rho_i(t,x)$, $i=1,2$ and $\rho_j(t,x)$, $j=3,4$ the traffic densities, respectively, on the incoming transmission lines and on the outgoing ones and by $(\rho_{1,0},\rho_{2,0},\rho_{3,0},\rho_{4,0})$ the initial datum.

Define $\gamma_i^{\max}$ and $\gamma_j^{\max}$ as in equations (4.3.3) and (4.3.4). This quantities represent the maximum flux that can be obtained by a single wave solution on each transmission line. In order to maximize the number of packets through the junction over incoming and outgoing lines we define

$$\Gamma = \min\left\{\Gamma_{in}^{\max}, \Gamma_{out}^{\max}\right\},$$

where $\Gamma_{in}^{\max} = \gamma_1^{\max} + \gamma_2^{\max}$ and $\Gamma_{out}^{\max} = \gamma_3^{\max} + \gamma_4^{\max}$. Thus we want to have Γ as flux through the junction.

From the results of Chapter 4, it is enough to determine the fluxes $\hat{\gamma}_i = f(\hat{\rho}_i)$, $i=1,2$. We have to distinguish two cases:

I $\Gamma_{in}^{\max} = \Gamma$,
II $\Gamma_{in}^{\max} > \Gamma$.

In the first case we set $\hat{\gamma}_i = \gamma_i^{\max}$, $i=1,2$, where the symbols $\hat{\gamma}_i$ denote the solution for the fluxes in the incoming transmission lines.
Let us analyze the second case in which we use the priority parameter q. Not

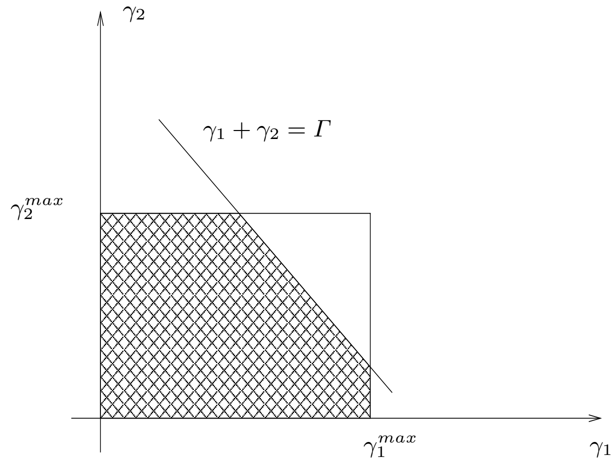


Fig. 9.3. Case $\Gamma_{in}^{\max} > \Gamma$.

all packets can enter the junction, so let C be the amount of packets that can go through. Then qC packets come from first incoming line and $(1-q)C$ packets from the second. This is the same as the precedence rule for car traffic, see Chapter 5.
In the plane (γ_1,γ_2) define the following lines:

$$r_q : \gamma_2 = \frac{1-q}{q}\gamma_1,$$

$$r_\Gamma : \gamma_1 + \gamma_2 = \Gamma.$$

Define P to be the point of intersection of the lines r_q and r_Γ. Recall that the final fluxes should belong to the region (see Figure 9.3):

$$\Omega = \left\{(\gamma_1, \gamma_2) : 0 \le \gamma_i \le \gamma_i^{\max}, i = 1, 2\right\}.$$

We distinguish two cases:

a) P belongs to Ω,
b) P is outside Ω.

In the first case we set $(\hat{\gamma}_1, \hat{\gamma}_2) = P$, while in the second case we set $(\hat{\gamma}_1, \hat{\gamma}_2) = Q$, with $Q = proj_{\Omega \cap r_\Gamma}(P)$ where *proj* is the usual projection on a convex set, see Figure 9.4.

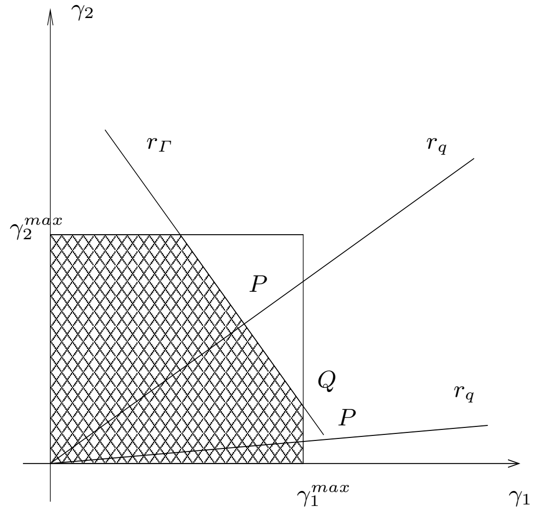


Fig. 9.4. P belongs to Ω and P is outside Ω.

The reasoning can be repeated also in the case of n incoming lines. In $\mathbb{R}^n$ the line r_q is given by $r_q = tv_q, t \in \mathbb{R}$, with $v_q \in \Delta_{n-1}$ where

$$\Delta_{n-1} = \left\{(\gamma_1, ..., \gamma_n) : \gamma_i \ge 0, i = 1, ..., n, \sum_{i=1}^{n} \gamma_i = 1\right\}$$

is the $(n-1)$ dimensional simplex and

$$H_\Gamma = \left\{ (\gamma_1, ..., \gamma_n) : \sum_{i=1}^{n} \gamma_i = \Gamma \right\}$$

is a hyperplane where $\Gamma = \min\{\sum_{in} \gamma_i^{\max}, \sum_{out} \gamma_j^{\max}\}$. Since $v_q \in \Delta_{n-1}$, there exists a unique point $P = r_q \cap H_\Gamma$. If $P \in \Omega$, then we set $(\hat{\gamma}_1, ..., \hat{\gamma}_n) = P$. If $P \notin \Omega$, then we set $(\hat{\gamma}_1, ..., \hat{\gamma}_n) = Q = proj_{\Omega \cap H_\Gamma}(P)$, the projection over the subset $\Omega \cap H_\Gamma$. Observe that the projection is unique since $\Omega \cap H_\Gamma$ is a closed convex subset of H_Γ.

Remark 9.3.1. A possible alternative definition in the case $P \notin \Omega$ is to set $(\hat{\gamma}_1, ..., \hat{\gamma}_n)$ as one of the vertices of $\Omega \cap H_\Gamma$.

Let us now determine $\hat{\gamma}_j, j = 3, 4$. As for the incoming transmission lines we have to distinguish two cases :

I $\Gamma_{out}^{\max} = \Gamma$,
II $\Gamma_{out}^{\max} > \Gamma$.

In the first case we again set $\hat{\gamma}_j = \gamma_j^{\max}, j = 3, 4$. Let us pass to the second case. Recall α the traffic distribution parameter. Since not all packets can go on the outgoing transmission lines, we let C be the amount that goes through. Then αC packets go on the outgoing line I_3 and $(1-\alpha)C$ on the outgoing line I_4. Notice that α plays the role of a traffic distribution coefficient.
Now we can proceed exactly as in the previous case with q replaced by α. More precisely, we define; r_α by the equation $\gamma_4 = \frac{1-\alpha}{\alpha}\gamma_3$, r_Γ by $\gamma_3 + \gamma_4 = \Gamma$ and P to be the point of intersection of the lines r_α and r_Γ. Setting: $\Omega = \{(\gamma_3, \gamma_4) : 0 \leq \gamma_j \leq \gamma_j^{\max}, j = 3, 4\}$, we distinguish two cases:

a) P belongs to Ω
b) P is outside Ω.

In the first case we set $(\hat{\gamma}_3, \hat{\gamma}_4) = P$, while in the second case we set $(\hat{\gamma}_3, \hat{\gamma}_4) = Q$, where $Q = proj_{\Omega \cap r_\Gamma}(P)$. Again, we can extend to the case of m outgoing lines as for the incoming lines defining the hyperplane $H_\Gamma = \{(\gamma_{n+1}, \ldots, \gamma_{n+m}) : \sum_{j=n+1}^{n+m} \gamma_j = \Gamma\}$ and choosing a vector $v_\alpha \in \Delta_{m-1}$.

Remark 9.3.2. An alternative way of choosing the vector v_α is the following. We assume that a traffic distribution matrix A is assigned, then we compute $\hat{\gamma}_1, ..., \hat{\gamma}_n$ as before and choose $v_\alpha \in \Delta_{m-1}$ by

$$v_\alpha = \Delta_{m-1} \cap \{tA(\hat{\gamma}_1, ..., \hat{\gamma}_n) : t \in \mathbb{R}\}.$$

The solution to Riemann problems in this section is consistent as shown by next Lemma.

Lemma 9.3.3. *The previous procedure defines a unique Riemann Solver.*

Proof. Let $\rho_0 = (\rho_{1,0}, \dots, \rho_{4,0})$ be the initial datum and $\hat{\rho} = RS(\rho_0)$. Assume, first, that $\Gamma < \Gamma_{in}^{\max}$. Define $\hat{\gamma}_i^{\max}$ to be the maximum flux on I_i given by a wave with left datum $\hat{\rho}_i$ and set then $\hat{\Gamma}_{in}^{\max} = \hat{\gamma}_1^{\max} + \hat{\gamma}_2^{\max}$. Then $\hat{\Gamma}_{in}^{\max} \geq \Gamma_{in}^{\max}$. Indeed if $\rho_{i,0} \in [0, \sigma[$ then $\hat{\rho}_i \in \{\rho_{i,0}\} \cup]\tau(\rho_{i,0}), \rho_{\max}]$ and $\hat{\gamma}_i^{\max} \geq \gamma_i^{\max} = f(\rho_{i,0})$. While if $\rho_{i,0} \in [\sigma, \rho_{\max}]$ then $\hat{\rho}_i \in [\sigma, \rho_{\max}]$ and so $\hat{\gamma}_i^{\max} = f(\sigma) = \gamma_i^{\max}$. The case $\Gamma < \Gamma_{out}^{\max}$ is treated similarly. □

9.4 Estimates on Density Variation

In this section we derive estimates on the total variation of the densities along a wave-front tracking approximate solution.

From now on, we assume that

(H1) every junction has exactly two incoming transmission lines and two outgoing ones.

This hypothesis is crucial, because the presence of more complicate junctions may provoke additional increases of the total variation of the flux and so of the density, as it was for car traffic see Chapter 5. The case where junctions have at most two incoming transmission lines and at most two outgoing ones can be treated in the same way.

From now on we fix a wave-front tracking approximate solution ρ, defined on the telecommunication network. Fix a junction J with two incoming transmission lines I_1 and I_2 and two outgoing ones I_3 and I_4. Suppose that at some time $\bar{t}$ a wave interacts with the junction J and let $(\rho_1^-, \rho_2^-, \rho_3^-, \rho_4^-)$ and $(\rho_1^+, \rho_2^+, \rho_3^+, \rho_4^+)$ indicate the equilibrium configurations at the junction J before and after the interaction respectively. Introduce the following notation

$$\gamma_i^{\pm} = f(\rho_i^{\pm}), \quad \Gamma_{in}^{\pm} = \gamma_{1,\max}^{\pm} + \gamma_{2,\max}^{\pm}, \quad \Gamma_{out}^{\pm} = \gamma_{3,\max}^{\pm} + \gamma_{3,\max}^{\pm},$$

$$\Gamma^{\pm} = \min\{\Gamma_{in}^{\pm}, \Gamma_{out}^{\pm}\},$$

where $\gamma_{i,\max}^{\pm}$, $i = 1, 2$ and $\gamma_{j,\max}^{\pm}$, $j = 3, 4$ are defined as in (4.3.3) and (4.3.4). In general $-$ and $+$ denote the values before and after the interaction, while by Δ we indicate the variation, i.e. the value after the interaction minus the value before. For example $\Delta\Gamma = \Gamma^+ - \Gamma^-$. Let us denote by $TV(f)^{\pm} = TV(f(\rho(\bar{t}\pm, \cdot)))$ the flux variation of waves before and after the interaction, and

$$TV(f)_{in}^{\pm} = TV(f(\rho_1(\bar{t}\pm, \cdot)) + TV(f(\rho_2(\bar{t}\pm, \cdot)),$$

$$TV(f)_{out}^{\pm} = TV(f(\rho_3(\bar{t}\pm, \cdot)) + TV(f(\rho_4(\bar{t}\pm, \cdot)),$$

the flux variation of waves before and after the interaction, respectively, on incoming and outgoing lines.

Let us prove some estimates which are used later to control the total variation of the density function. For simplicity, from now on we assume that:

(A) the wave interacting at time $\bar{t}$ with J comes from line I_1 and we let ρ_1 be the value on the left of the wave.

The case of a wave from an outgoing line can be treated similarly.

Lemma 9.4.1. *We have*

$$sgn\,(\Delta\gamma_3) \cdot sgn\,(\Delta\gamma_4) \geq 0.$$

Proof. To prove the lemma it is enough to observe that a variation of γ_3 is due to a movement along the line r_q or along $\gamma_3 = c_1$ or $\gamma_4 = c_2$ with c_1 and c_3 constant. In each case $\Delta\gamma_3$ and $\Delta\gamma_4$ have the same sign. □

In the same way we can prove the following Lemma:

Lemma 9.4.2. *We have*

$$sgn(\gamma_1^+ - \gamma_1) \cdot sgn(\Delta\gamma_2) \geq 0,$$

where $\gamma_1 = f(\rho_1)$.

Lemma 9.4.3. *It holds*

$$TV(f)^+_{out} = |\Delta\Gamma|.$$

Proof. To prove the lemma it is enough to observe that

$$\Gamma^- = \gamma_3^- + \gamma_4^-, \quad \Gamma^+ = \gamma_3^+ + \gamma_4^+,$$

$$|\Delta\Gamma| = |\Gamma^+ - \Gamma^-| = |(\gamma_3^+ - \gamma_3^-) + (\gamma_4^+ - \gamma_4^-)|$$

from which, by Lemma 9.4.1, we have

$$|\Delta\Gamma| = |\Delta\gamma_3| + |\Delta\gamma_4| = TV(f)^+_{out}.$$

This completes the proof. □

Lemma 9.4.4. *We have*

$$TV(f)^-_{in} = TV(f)^+_{in} + |\Delta\Gamma|. \tag{9.4.4}$$

Proof. Clearly since the wave on the first line has positive velocity, we have $0 \leq \rho_1 \leq \sigma$. Since $\rho_1 \leq \sigma$, observe that the maximum flux for ρ_1^+, which is the solution with initial data ρ_1, is given by $\gamma_{1,\max} = f(\rho_1)$. Also

$$TV(f)^- = TV(f)^-_{in} = |\gamma_1 - \gamma_1^-|.$$

We have two possibilities:

Case 1) $\rho_1^- \leq \sigma$,
Case 2) $\rho_1^- > \sigma$.

Let us first analyze Case 1). Then we further split it into two subcases:

Case 1a) $\rho_1 < \rho_1^-$,
Case 1b) $\rho_1 > \rho_1^-$.

If 1a) holds true, since $\rho_1 < \rho_1^-$, we get $\gamma_{1,\max} = f(\rho_1) < f(\rho_1^-) = \gamma_{1,\max}^-$ and one of the following holds:

Case 1a.1) $\Gamma^- = \Gamma_{in}^-$,
Case 1a.2) $\Gamma^- = \Gamma_{out}^-$.

In Case 1a.1) from $\gamma_{1,\max} < \gamma_{1,\max}^-$ and $\Gamma^- = \Gamma_{in}^-$, it follows that $\Gamma^+ = \Gamma_{in}^+$, from which $\gamma_2^+ = \gamma_2^-$, $\gamma_1^+ = \gamma_1$ and then $TV(f)_{in}^+ = 0$.

In the other Case 1a.2) we have $\gamma_{1,\max} < \gamma_{1,\max}^-$, hence $\Gamma_{in}^- \geq \Gamma^-$ and $\gamma_{1,\max} + \gamma_{2,\max}^- < \Gamma_{in}^-$. The following distinction must be considered:

Case 1a.2.1) $\gamma_{1,\max} + \gamma_{2,\max}^- \geq \Gamma^-$,
Case 1a.2.2) $\gamma_{1,\max} + \gamma_{2,\max}^- < \Gamma^-$.

If Case 1a.2.1) holds, from $\gamma_{1,\max} + \gamma_{2,\max}^- \geq \Gamma^-$, we have that $\Gamma^+ = \Gamma^-$, from which $|\Delta\Gamma| = 0$. By Lemma 9.4.2 the conclusion holds.

In the opposite Case 1a.2.2) from $\gamma_{1,\max} + \gamma_{2,\max}^- < \Gamma^-$, one gets $\Gamma^+ = \gamma_{1,\max} + \gamma_{2,\max}^-$, from which it follows that $\mathrm{TV}(f)_{in}^+ = 0$. Then $|\Delta\Gamma| = \left|\gamma_1^- - \gamma_1\right| = TV(f)_{in}^-$. Case 1a) is thus finished.

Let us now focus on Case 1b). We have to distinguish two possibilities:

Case 1b.1) $\Gamma^- = \Gamma_{out}^-$,
Case 1b.2) $\Gamma^- = \Gamma_{in}^-$.

If Case 1.b.1) holds, from $\Gamma^- = \Gamma_{out}^-$ it follows that $\gamma_{1,\max} + \gamma_{2,\max}^- > \Gamma_{in}^-$. Then $\Gamma^+ = \Gamma^-$, hence $|\Delta\Gamma| = 0$ and by Lemma 9.4.2 the conclusion holds.

In Case 1.b.2), we have $\gamma_{1,\max} + \gamma_{2,\max}^- > \Gamma_{in}^-$ and $\Gamma_{out}^- \geq \Gamma_{in}^-$ and following cases may happen:

Case 1b.2.1) $\gamma_{1,\max} + \gamma_{2,\max}^- \leq \Gamma_{out}^-$,
Case 1b.2.2) $\gamma_{1,\max} + \gamma_{2,\max}^- > \Gamma_{out}^-$.

Consider Case 1b.2.1) first. From $\gamma_{1,\max} + \gamma_{2,\max}^- \leq \Gamma_{out}^-$, one has $TV(f)_{in}^+ = 0$, hence $|\Delta\Gamma| = \left|\gamma_1 - \gamma_1^-\right| = TV(f)_{in}^-$.

In Case 1b.2.2), from $\gamma_{1,\max} + \gamma_{2,\max}^- > \Gamma_{out}^-$ we obtain $\Gamma^+ = \Gamma_{out}^+$. By Lemma 9.4.1,

$$TV(f)_{in}^+ = \gamma_{1,\max} + \gamma_{2,\max}^- - \Gamma_{out}^-,$$

$$TV(f)_{in}^- = \gamma_{1,\max} - \gamma_{1,\max}^-,$$

hence

$$TV(f)_{in}^- - TV(f)_{in}^+ = -\gamma_{1,\max}^- - \gamma_{2,\max}^- + \Gamma_{out}^- = \\ = \Gamma^+ - \Gamma_{in}^- = \Gamma^+ - \Gamma^- = |\Delta\Gamma|\,.$$

Let us analyze Case 2). Since $\rho_1^- > \sigma$ it follows that $\rho_1 < \tau(\rho_1^-) < \sigma$. Observe that $\gamma_1 = f(\rho_1) < f(\rho_1^-) = \gamma_1^-$ and $\gamma_{1,\max}^- = f(\sigma), \gamma_{1,\max} = f(\rho_1)$.

We have to distinguish two cases:

Case 2.a) $\Gamma^- = \Gamma_{in}^-$,
Case 2.b) $\Gamma^- = \Gamma_{out}^-$.

If Case 2.a) holds, then one gets $\gamma_{1,\max} + \gamma_{2,\max}^- < \Gamma^-$, from which it follows that $\Gamma^+ = \gamma_{1,\max} + \gamma_{2,\max}^-$. Hence $TV(f)_{in}^+ = 0$ and the conclusion holds.

For the opposite Case 2.b), we have $\gamma_{1,\max} + \gamma_{2,\max}^- < \Gamma_{in}^-$ and $\Gamma_{in}^- \geq \Gamma_{out}^-$. Hence the following two cases are possible:

Case 2.b.1) $\gamma_{1,\max} + \gamma_{2,\max}^- \geq \Gamma_{out}^-$,
Case 2.b.2) $\gamma_{1,\max} + \gamma_{2,\max}^- < \Gamma_{out}^-$.

In Case 2.b.1), from $\gamma_{1,\max} + \gamma_{2,\max}^- \geq \Gamma_{out}^-$, it follows that $\Gamma^+ = \Gamma^-$. The latter implies $|\Delta\Gamma| = 0$ and the conclusion follows from Lemma 9.4.2.

In Case 2.b.2) from $\gamma_{1,\max} + \gamma_{2,\max}^- < \Gamma_{out}^-$, we obtain $\Gamma^+ = \gamma_{1,\max} + \gamma_{2,\max}^-$. Thus, by Lemma 9.4.2, we get:

$$TV(f)_{out}^+ = \Gamma^+ - (\gamma_1 + \gamma_2^-) = (\gamma_{1,\max} + \gamma_{2,\max}^-) - (\gamma_1 + \gamma_2^-) = \\ = \gamma_{2,\max}^- - \gamma_2^- .$$

It follows that

$$|\Delta\Gamma| = \Gamma^- - \Gamma^+ = \gamma_1^- + \gamma_2^- - (\gamma_{1,\max} + \gamma_{2,\max}^-) \\ = (\gamma_1^- - \gamma_{1,\max}) + (\gamma_2^- - \gamma_{2,\max}^-) = TV(f)_{in}^- - TV(f)_{out}^+,$$

and the conclusion holds. The proof is thus finished. □

From the above results, we obtain the following:

Lemma 9.4.5. *The flux variation $TV(f)$ is conserved along wave-front tracking approximations for interactions of waves with a junction.*

Proof. From Lemma 9.4.3 and Lemma 9.4.4 we get

$$TV(f)^- = TV(f)_{in}^- = TV(f)_{in}^+ + |\Delta\Gamma| = TV(f)^+.$$

The proof is finished. □

We can now define big shocks as in 4.3.4, bad datum as in 4.3.5, and BEFs as in Definition 5.3.8 Our aim is now to bound, for each line I_i, the number of big waves inside the region $D_2^i(\rho)$, i.e. those generated by the influence of external lines.

Lemma 9.4.6. *Let $\bar{t}$ be the time at which the two BEFs $Y_{\pm}^{i,\rho}$ interact. Assume $\bar{t} < +\infty$, $Y_{\pm}^{i,\rho}(\bar{t}) \in]a_i, b_i[$ and define*

$$\hat{\rho}_{out} = \rho\left(Y_{\pm}^{i,\rho}(\bar{t})-\right), \quad \hat{\rho}_{in} = \rho\left(Y_{\pm}^{i,\rho}(\bar{t})+\right),$$
$$\rho^* = \lim_{t \uparrow \bar{t}} \rho\left(Y_{-}^{i,\rho}(t)+\right) = \lim_{t \uparrow \bar{t}} \rho\left(Y_{+}^{i,\rho}(t)-\right).$$

If $\hat{\rho}_{in}$, respectively $\hat{\rho}_{out}$, is a bad datum for I_i as incoming line, respectively for I_i as outgoing line, then there exists no value ρ^* of the density such that

$$\lambda(\hat{\rho}_{out}, \rho^*) > \lambda(\rho^*, \hat{\rho}_{in}).$$

Proof. Since $\hat{\rho}_{out}$ and $\hat{\rho}_{in}$ are bad data for, respectively, an outgoing transmission line and an incoming transmission line, it follows that

$$\hat{\rho}_{out} \in]\sigma, 1], \quad \hat{\rho}_{in} \in [0, \sigma[.$$

Observe that $\hat{\rho}_{out}$ and ρ^* must be connected by a single wave, thus $\rho^* \geq \sigma$, otherwise the wave would be split in a fan of rarefaction shocks.
Similarly, ρ^* and $\hat{\rho}_{in}$ must be connected by a single wave, thus $\rho^* \leq \sigma$, otherwise the wave would be split in a fan of rarefaction shocks.
Finally, $\rho^* = \sigma$, but then

$$\lambda(\hat{\rho}_{out}, \rho^*) \leq 0 \leq \lambda(\rho^*, \hat{\rho}_{in})$$

and the conclusion holds. □

Lemma 9.4.7. *For every $t \geq 0$, there are at most two big waves on*

$$\left\{x : (t, x) \in D_2^i(\rho)\right\} \subseteq [a_i, b_i].$$

Proof. A big wave can originate at time t on transmission line I_i from J only if the line I_i has a bad datum at J at time t. If this happens, then, from Lemma 4.3.6, line I_i has not a bad datum at J up to the time in which a big wave is absorbed from I_i. This concludes the proof if $D_2^i(\rho)$ is formed by two connected components.

It remains to consider the time at which the two BEFs interact. By Lemma 9.4.6 we have that not both connected components can contain a big wave. Thus again there are at most two big waves. □

The obtained estimates on the number of big waves permits to bound the total variation of the densities as follows.

Theorem 9.4.8. *Consider a telecommunication network $(\mathcal{I}, \mathcal{J})$, assume (F), (H1) and consider the Riemann solver of Section 9.3. Let ρ be a wave-front tracking approximate solution, then*

$$TV(\rho(t,\cdot)) \leq TV(\rho(0,\cdot)) + 2N\left(\frac{f(\sigma)}{\bar{v}} + 1\right),$$

for each $t \geq 0$, where N is the total number of transmission lines of the network. Moreover given $T > 0$, there exists an admissible solution to the Cauchy problem on the network defined on $[0,T]$ for every initial data.

Proof. Let $TV(h;[a,b])$ denote the total variation of the function h over the interval $[a,b]$ and define

$$TV^j(\rho(t)) = \sum_i TV(\rho(t); D^i_j(\rho(t))), \qquad j = 1, 2,$$

which are, respectively, the total variation of $\rho(t)$ due to the evolution only inside each line I_i and by interaction with junctions. Clearly:

$$TV(\rho(t)) = TV^1(\rho(t)) + TV^2(\rho(t)).$$

Since $D^i_1(\rho(t))$ is not influenced by external lines, we are in the situation of a conservation law on $\mathbb{R}$, hence

$$TV^1(\rho(t)) \leq TV(\rho(0)).$$

Let $B(t)$ denote the number of big waves generated from junctions, i.e. the number of big waves in $\cup_i D^i_2(\rho(t))$. Then by chain rule for BV functions:

$$TV^2(\rho(t)) \leq \frac{1}{\bar{v}} TV^2(f(\rho(t)) + B(t) \leq \frac{1}{\bar{v}}(TV^2(f(\rho(0+))) + B(t). \tag{9.4.5}$$

Now $TV^2(\rho(0)) = 0$, thus, using the previous Lemmas, the following relation holds:

$$TV^2(\rho(t)) \leq \frac{1}{\bar{v}} 2Nf(\sigma) + 2N, \tag{9.4.6}$$

and the estimate on $TV(\rho)$ holds.

The second part of the statement follows from Theorem 2.5.4 □

9.5 Uniqueness and Lipschitz Continuous Dependence

The same approach of Section 2.7 can be used on networks. There are various alternative methods to treat uniqueness and continuous dependence for the case of conservation laws on the real line, see the bibliographical note to Chapter 2. No one of these methods seems to work for the network case. In fact, Kruzkov method requires to estimate integrals on a region in $\mathbb{R}^2$, which now is replaced by an integral on the topological space obtained by the product of the network and $\mathbb{R}$. On the other side, it is not clear how to define a viscous solutions on the network, in particular how to treat boundary data at nodes, and how to pass to the limit. Finally, a Bressan-Liu-Yang type functional requires to introduce a definition of approaching waves, but, on a general network, with complicate topology, every wave is potentially approaching each other.

Let us now go back to the network case. If $\rho = (\rho_1, ..., \rho_N)$ is a solution on the network then we set

$$\|\rho\|_{L^1} = \sum_i \|\rho_i\|_{L^1(I_i)}.$$

To estimate the distance among wave-front tracking solutions it is thus enough to prove (2.7.49). We prove the latter estimating the evolution of the tangent vector norm at each time. For this, we fix a time $\bar{t} \geq 0$ and, without loss of generality, treat the following cases:

a) no interaction of waves takes place in any transmission line at $\bar{t}$ and no wave interacts with a junction;
b) two waves interact at $\bar{t}$ on a transmission line and no other interaction takes place;
c) a wave interacts with a junction at $\bar{t}$ and no other interaction takes place.

Case a) can be treated as Case 1. of Section 2.7.

Case b) can be treated as Case 2. of Section 2.7, using Lemma 2.7.2.

For Case c), recall Lemma 5.4.1 of Chapter 5. Define $TV(f)^{\pm}$ to be the total variation of the flux of the solution before $(-)$ and after $(+)$ the interaction, and $TV(f)_i^{\pm}$ the same quantity on line I_i. Without loss of generality, we can assume that a wave from an incoming transmission line $\bar{\imath}$ interacts with a junction J and no other wave is present. Then $TV(f)^- = TV(f)_{\bar{\imath}}^-$ and $TV(f)^+ = \sum_j TV(f)_j^+$ where $TV(f)_j^+$ measures just the wave produced by the interaction. From Lemma 2.7.2 we have

$$|\xi_j|\,|\Delta\rho_j| = \frac{TV(f)_j^-}{TV(f)_{\bar{\imath}}^-}\,|\xi_{\bar{\imath}}|\,|\Delta\rho_{\bar{\imath}}|\,.$$

Using Lemma 9.4.5 we conclude

$$\begin{aligned} \left\|(v,\xi)^+\right\| &= \sum_j |\xi_j|\,|\Delta\rho_j| = \sum_j \frac{TV(f)_j^-}{TV(f)_{\bar{\imath}}^-}\,|\xi_{\bar{\imath}}|\,|\Delta\rho_{\bar{\imath}}| \\ &= \frac{TV(f)^+}{TV(f)^-}\left\|(v,\xi)^-\right\| = \left\|(v,\xi)^-\right\|. \end{aligned} \tag{9.5.7}$$

From (2.7.50), (2.7.53) and (9.5.7), we get the following:

Theorem 9.5.1. *Consider a telecommunication network $(\mathcal{I}, \mathcal{J})$, assume (H1) and consider the Riemann solver of Section 9.3. Then the solutions to Cauchy problems on the networks are unique and depend in a Lipschitz continuous way from initial data.*

10

Numerics on Networks

In this chapter we develop some numerical algorithms to simulate the behavior of the urban traffic flow. We focus on the Lighthill-Whitham-Richards model on each road network and, at junctions, on the Riemann solver proposed in Chapter 5.

10.1 Numerical Approximation

For definitiveness, fix the following flux

$$f(\rho) = v_{max}\ \rho \left(1 - \frac{\rho}{\rho_{max}}\right), \tag{10.1.1}$$

thus, setting for simplicity $\rho_{max} = 1 = v_{max}$,

$$f(\rho) = \rho(1 - \rho).$$

The maximum $\sigma = 1/2$ is then unique: $f(\sigma) = \max_{[0,1]} f(\rho) = f_M$.

We define a **numerical grid** in $\mathbb{R}^N \times (0, T)$ using the following notations:

- Δx is the space grid size;
- Δt is the time grid size;
- $(x_m, t_n) = (m\Delta x, n\Delta t)$ for $n \in \mathbb{N}$ and $m \in \mathbb{Z}$ are the grid points.

For a function v defined on the grid we write $v_m^n = v(x_m, t_n)$ for m, n varying on a subset of $\mathbb{Z}$ and $\mathbb{N}$ respectively. We also use the notation u_m^n for $u(x_m, t_n)$ when u is a continuous function on the (x, t) plane.

10.1.1 Godunov Scheme

A good numerical method to solve the equations along roads is represented by the Godunov scheme, which is based on exact solutions to Riemann problems;

see [51, 52]. The idea is the following: first the initial datum is approximated by a piecewise constant function; then the corresponding Riemann problems are solved exactly and a global solution is simply obtained by piecing them together; finally, one takes the mean and proceeds by induction.
Let us now consider in detail the scheme. We take an approximation of the initial datum u_0 with the following mean values:

$$v_m^0 = \frac{1}{\Delta x}\int_{x_{m-\frac{1}{2}}}^{x_{m+\frac{1}{2}}} u_0(x)dx, \quad m \in \mathbb{Z}. \tag{10.1.2}$$

Godunov scheme is based on exact solutions v^Δ to Riemann problems at points $(m-\frac{1}{2})\Delta x$, $m \in \mathbb{Z}$ and then on the projection of the solution

$$v_m^{n+1} = \frac{1}{\Delta x}\int_{x_{m-\frac{1}{2}}}^{x_{m+\frac{1}{2}}} v^\Delta(t_{n+1}, x)dx. \tag{10.1.3}$$

This procedure can be repeated inductively on every t_n. Under the CFL condition

$$\Delta t \sup_{m,n}\left\{\sup_{u\in I(u_m^n, u_{m+1}^n)} |f'(u)|\right\} \leqslant \Delta x, \tag{10.1.4}$$

the waves, generated by different Riemann problems do not interact. We can use the Gauss-Green formula to compute v^{n+1} and the flux in $x = x_m - \frac{1}{2}\Delta x$ for $t \in (t_n, t_{n+1})$ is given by $f(u(t, x_m - \frac{1}{2}\Delta x)) = f(W_R(0; v_{m-1}^n, v_m^n))$, where $W_R\left(\frac{x}{t}; v_-, v_+\right)$ is the self-similar solution between v_- and v_+. Similarly for the point $x = x_m + \frac{1}{2}\Delta x$: $f(u(t, x_m + \frac{1}{2}\Delta x)) = f(W_R(0; v_m^n, v_{m+1}^n))$. As the flux is time invariant and continuous, we can put it out of the integral and, setting $g^G(u, v) = f(W_R(0; u, v))$ under the condition (10.1.4), the scheme can be written as:

$$v_m^{n+1} = v_m^n - \frac{\Delta t}{\Delta x}\big(g^G(v_m^n, v_{m+1}^n) - g^G(v_{m-1}^n, v_m^n)\big). \tag{10.1.5}$$

The expression of the numerical flux for Godunov method is in general given by

$$g^G(u, w) = \begin{cases} \min_{z\in[u,w]} f(z), & \text{if} \quad u \leqslant w, \\ \max_{z\in[w,u]} f(z), & \text{if} \quad w \leqslant u. \end{cases} \tag{10.1.6}$$

10.1.2 Kinetic Method for a Boundary Value Problem

Here we present the kinetic scheme for initial-boundary value conservation equations:

$$u_t + F(u)_x = 0, \tag{10.1.7}$$
$$u(x, 0) = u_0(x), \quad x \geqslant 0, \tag{10.1.8}$$
$$u(0, t) = u_b(t), \quad t \geqslant 0, \tag{10.1.9}$$

and (10.1.9) can be imposed only where it is compatible with the trace of the solution to the problem and with the flux F. We have $u(x,t) \in \mathbb{R}$ for $x \geqslant 0$, $t \geqslant 0$, and F is a Lipschitz continuous function.

A kinetic approximation of the problem (10.1.7) is obtained solving the following BGK-like system of N non-linear equations:

$$\partial_t f_k^\varepsilon + \lambda_k \partial_x f_k^\varepsilon = \frac{1}{\varepsilon}\left(M_k(u^\varepsilon) - f_k^\varepsilon\right), \tag{10.1.10}$$

where the λ_k are fixed velocities (a set of real numbers not all zero), ε is a positive parameter, and each f_k^ε is a function of $\mathbb{R}^+ \times [0,T]$ with values in $\mathbb{R}$. We impose the corresponding initial and boundary data:

$$f_k^\varepsilon(x,0) = M_k(u_0(x)), \quad x \in \mathbb{R}^+, \tag{10.1.11}$$

$$f_k^\varepsilon(0,t) = M_k(u_b(t)) \quad \forall \lambda_k > 0 \text{ and } t \geqslant 0. \tag{10.1.12}$$

Functions M_k, $k = 1,\ldots,N$, are called Maxwellian functions. To have the convergence of $u^\varepsilon = \sum_{k=1}^N f_k^\varepsilon$ when $\varepsilon \to 0$ towards the solution of the problem (10.1.7), we need to impose the compatibility conditions:

$$\sum_{k=1}^N M_k(u) = u, \qquad \sum_{k=1}^N \lambda_k M_k(u) = F(u). \tag{10.1.13}$$

A sufficient condition for convergence is represented by the monotonicity condition MND

$$\min_k \lambda_k \leqslant F'(u) \leqslant \max_k \lambda_k. \tag{10.1.14}$$

Kinetic Approximations

We describe some different approximations of kinetic schemes.

- **Two velocities model.** Take $N = 2$ and $\lambda_1 = -\lambda_2 = -\lambda$. We approximate the scalar conservation law

  $$\rho_t + f(\rho)_x = 0$$

  by a relaxation system which is diagonalized in the form

  $$\begin{cases} \partial_t f_1^\varepsilon - \lambda \partial_x f_1^\varepsilon = \frac{1}{\varepsilon}(M_1(u^\varepsilon) - f_1^\varepsilon) \\ \partial_t f_2^\varepsilon + \lambda \partial_x f_2^\varepsilon = \frac{1}{\varepsilon}(M_2(u^\varepsilon) - f_2^\varepsilon). \end{cases}$$

  The associated Maxwellian functions are

  $$M_1(u) = \frac{1}{2}\left(u - \frac{f(u)}{\lambda}\right), \quad M_2(u) = \frac{1}{2}\left(u + \frac{f(u)}{\lambda}\right).$$

  In order to respect the monotonicity condition MND, we deduce the following relation for the velocity vector λ:

  $$\max |f'(u)| < \lambda. \tag{10.1.15}$$

- **Three velocities model.** Take $N = 3$ and $\lambda_3 = -\lambda_1 = \lambda > 0$, $\lambda_2 = 0$. The approximated kinetic system has the Maxwellian functions given by

$$M_1(u) = \frac{1}{\lambda}\begin{cases} 0, & \text{if } u \leqslant \frac{1}{2}, \\ u(u-1) + \frac{1}{4}, & \text{if } u \geqslant \frac{1}{2}, \end{cases}$$

$$M_2(u) = \begin{cases} \left(1 - \frac{1}{\lambda}\right) u + \frac{1}{\lambda}u^2, & \text{if } u \leqslant \frac{1}{2}, \\ \left(1 + \frac{1}{\lambda}\right) u - \frac{1}{\lambda}u^2 - \frac{1}{2\lambda}, & \text{if } u \geqslant \frac{1}{2}, \end{cases}$$

$$M_3(u) = \frac{1}{\lambda}\begin{cases} u(1-u), & \text{if } u \leqslant \frac{1}{2}, \\ \frac{1}{4}, & \text{if } u \geqslant \frac{1}{2}. \end{cases}$$

At the boundary we impose $f_3(0,t) = M_3(u_b(t))$ and the monotonicity condition holds if and only if the condition (10.1.15) is satisfied. In this case (10.1.15) reads

$$0 \leqslant M_2'(u) \leqslant 1 - \frac{|f'(u)|}{\lambda}.$$

Numerical Scheme

Following [6, 7], we discretize the problem (10.1.10)-(10.1.11)-(10.1.12) and making ε tend to zero, we obtain a numerical scheme for the initial boundary value problem for the conservation law (10.1.7). As usual, we discretize data of the problem by a piecewise constant approximation and we take for all k:

$$\begin{aligned} f^n_{-1,k} &= M_k(u^n_b), \quad 0 \leqslant n \leqslant M-1, \\ f^0_{m,k} &= M_k(u^0_m), \quad m \in \mathbb{N}. \end{aligned}$$

The operators used to solve system (10.1.10) are splitted into the *transport* part and the *collision* part.

For the transport contribute, the scheme written in the Harten formulation including both first and second order in space approximation reads:

$$m \geqslant 0, \begin{cases} f^{n+\frac{1}{2}}_{m,k} = f^n_{m,k}(1 - D^n_{i-\frac{1}{2},k}) + D^n_{i-\frac{1}{2},k} f^n_{i-1,k}, & \text{if } \lambda_k > 0, \\ f^{n+\frac{1}{2}}_{m,k} = f^n_{m,k}(1 - D^n_{i+\frac{1}{2},k}) + D^n_{i+\frac{1}{2},k} f^n_{i+1,k}, & \text{if } \lambda_k \leqslant 0. \end{cases} \tag{10.1.16}$$

Note that it is necessary to assign the boundary value $f^n_{b,k} = f^n_{-1,k}$ only for positive velocities. A first order in space upwind approximation is chosen:

$$D^n_{i-\frac{1}{2},k} = D^n_{i+\frac{1}{2},k} = \xi_k = |\lambda_k| \frac{\Delta t}{\Delta x}$$

and in that case (10.1.16) is well defined even for $m = 0$.

The transport part can be approximated by a second order scheme as follows. Starting from $f^n_{m,k}$ we build a piecewise linear function:

$$\bar{f}^n_{m,k}(x) = f^n_{m,k} + (x - x_m)\sigma^n_{m,k}, \quad x \in (x_{m-\frac{1}{2}}, x_{m+\frac{1}{2}}),$$

where $\sigma^n_{m,k}$ are limited slopes and we solve exactly the transport equations on $[t_n, t_{n+1}]$. Projecting the solution on the set of piecewise constant functions on the cells, we obtain the explicit expression for $D^n_{m+\frac{1}{2},k}$:

$$D^n_{m+\frac{1}{2},k} = \xi_k \left(1 + \text{sgn }(\lambda_k)\Delta x \frac{(1-\xi_k)}{2} \frac{(\sigma^n_{m+1,k} - \sigma^n_{m,k})}{\Delta f^n_{m+\frac{1}{2},k}}\right), \tag{10.1.17}$$

with the convention that if $\Delta f^n_{m+\frac{1}{2},k} = 0$, then $D^n_{m+\frac{1}{2},k} = \xi_k = |\lambda_k|\frac{\Delta t}{\Delta x}$. Note that if $\lambda_k > 0$, then (10.1.17) is defined for $m \geqslant -1$, in the other cases is available for $m \geqslant 0$. The slopes $\sigma^n_{m,k}$ for $m \geqslant 1$ are:

$$\sigma^n_{m,k} = \text{minmod}\left(\frac{\Delta f^n_{m+\frac{1}{2},k}}{\Delta x}, \frac{\Delta f^n_{m-\frac{1}{2},k}}{\Delta x}\right),$$

with $\Delta f^n_{m+\frac{1}{2},k} = f^n_{m+1,k} - f^n_{m,k}$ and $\text{minmod}(a,b) = \min(|a|,|b|)\frac{sgn(a)+sgn(b)}{2}$. For the convergence results see [6]. The time step restriction for both cases is

$$\max_{1 \leqslant k \leqslant N} |\lambda_k| \Delta t \leq \Delta x. \tag{10.1.18}$$

Then we use the solution obtained from the precedent scheme as the initial condition for collision system. Under the compatibility conditions (10.1.13) we find the exact solution of the system, that for $\varepsilon \to 0$ is

$$f^{n+1}_{m,k} = M_k(u^{n+\frac{1}{2}}_m) = M_k(u^{n+1}_m), \quad m \geqslant 0, \quad n \geqslant 1 \tag{10.1.19}$$

and the identity holds

$$u^{n+1}_m = \sum_k f^{n+\frac{1}{2}}_{m,k} = u^{n+\frac{1}{2}}_m. \tag{10.1.20}$$

Assuming the monotonicity condition, we deduce the usual CFL condition

$$\max_u |f'(u)| \Delta t \leq \Delta x$$

and, from the transport part of the scheme, we have to impose the time step restriction in (10.1.18).

10.1.3 Boundary Conditions and Conditions at Junctions

Let us start with the Godunov scheme.

- **Boundary conditions.** Suppose to assign a condition at the incoming boundary $x = 0$:
$$u(0,t) = \rho_1(t), \quad t > 0$$
and study equation only for $x > 0$. We are considering the initial-boundary value problem (10.1.7)-(10.1.8)-(10.1.9). It is not easy to find a function u that satisfies (10.1.9) in a classical sense, because, in general, the boundary data cannot be assumed. One seeks a condition which is to be effective only in the inflow part of the boundary. Following [15], the rigorous way to assign the boundary condition is:
$$\max_{k \in I(u(0,t),\rho_1(t))} \Big\{ \mathrm{sgn}\ (u(0,t) - \rho_1(t))[F(u(0,t)) - F(k)] \Big\} = 0. \quad (10.1.21)$$
We proceed by inserting a ghost cell and defining
$$v_0^{n+1} = v_0^n - \frac{\Delta t}{\Delta x}\big(g^G(v_0^n, v_1^n) - g^G(u_1^n, v_0^n)\big), \quad (10.1.22)$$
where
$$u_1^n(t) = \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} \rho_1(t)dt$$
takes the place of v_{-1}^n. An outgoing boundary can be treated analogously. Let $x < L = x_N$. Then the discretization reads:
$$v_N^{n+1} = v_N^n - \frac{\Delta t}{\Delta x}\big(g^G(v_N^n, u_2^n) - g^G(v_{N-1}^n, v_N^n)\big), \quad (10.1.23)$$
where
$$u_2^n(t) = \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} \rho_2(t)dt$$
takes the place of v_{N+1}^n, that is a ghost cell value.
- **Conditions at a junction.** For roads connected to a junction at the right endpoint we set
$$v_N^{n+1} = v_N^n - \frac{\Delta t}{\Delta x}\big(\hat{\gamma}_i - g^G(v_{N-1}^n, v_N^n)\big),$$
while for roads connected to a junction at the right endpoint we have
$$v_0^{n+1} = v_0^n - \frac{\Delta t}{\Delta x}\big(g^G(v_0^n, v_1^n) - \hat{\gamma}_j\big),$$
where $\hat{\gamma}_i, \hat{\gamma}_j$ are the maximized fluxes.

Remark 10.1.1. For the Godunov's scheme there is no need to invert the flux f to put it in the scheme, as the Godunov's flux coincides with the Riemann's flux. In this case it suffices to insert the computed maximized fluxes directly in the scheme.

Let us analyse now the case of Kinetic schemes.

- **Boundary conditions.** For $m = 0$ we take for the boundary

$$\sigma^n_{-1,k} = 0.$$

In this case, the slope $\sigma^n_{0,k}$ can be defined as

1. for $\lambda_k > 0$:

$$\sigma^n_{0,k} = \text{minmod}\left(\frac{f^n_{1,k} - f^n_{0,k}}{\Delta x}, 2\frac{f^n_{0,k} - M_k(u^n_b)}{\Delta x}\right),$$

where u^n_b is the boundary condition;

2. for $\lambda_k < 0$:

$$\sigma^n_{0,k} = \frac{f^n_{1,k} - f^n_{0,k}}{\Delta x}.$$

When $m = N$, the scheme for $\lambda_k < 0$ requires the values $f^n_{N+1,k}$, $f^n_{N+2,k}$, that can be obtained, for instance, by imposing a Neumann condition.

- **Conditions at a junction.** As usual, to impose the boundary condition at a junction we need to examine the links between the roads. At the right boundary ($m = N$) of roads linked to the junction on the right endpoint one has:

$$f^{n+\frac{1}{2}}_{N,k} = f^n_{N,k}(1 - D^n_{N+\frac{1}{2},k}) + D^n_{N+\frac{1}{2},k} f^n_{N+1,k}, \qquad \text{for } \lambda_k < 0,$$

with

$$f^n_{N+1,k} = M_k(f^{-1}(\hat{\gamma}_i)).$$

Moreover we use the Neumann condition $f^n_{N+2,k} = f^n_{N+1,k}$ for roads which are not linked to the junction on the right. At the left boundary ($m = 0$) of roads linked to the junction on the left endpoint the scheme in case $\lambda_k > 0$ reads:

$$f^{n+\frac{1}{2}}_{0,k} = f^n_{0,k}(1 - D^n_{-\frac{1}{2},k}) + D^n_{-\frac{1}{2},k} f^n_{-1,k},$$

with

$$f^n_{-1,k} = M_k(f^{-1}(\hat{\gamma}_j)).$$

Notice that $\hat{\gamma}_i$, $\hat{\gamma}_j$ are the maximized incoming and outgoing fluxes, where the inversion of the flux function f follows the rules

1. for roads entering the junction:
   a) if $u^n_N \in [0, \sigma]$ and $\hat{\gamma}_i < F(u^n_N)$ then $F^{-1}(\hat{\gamma}_i) \in [\tau(u^n_N), 1)$,
   b) if $u^n_N \in [0, \sigma]$ and $\hat{\gamma}_i = F(u^n_N)$ then $F^{-1}(\hat{\gamma}_i) = u^n_N$,
   c) if $u^n_N \in [\sigma, 1]$ then $F^{-1}(\hat{\gamma}_i) \in [\sigma, 1]$,
   with $i = 1, 2$.
2. for roads coming out of the junction:
   a) if $u^n_0 \in [\sigma, 1]$ and $\hat{\gamma}_j < F(u^n_0)$ then $F^{-1}(\hat{\gamma}_j) \in [0, \tau(u^n_0))$,
   b) if $u^n_0 \in [\sigma, 1]$ and $\hat{\gamma}_j = F(u^n_0)$ then $F^{-1}(\hat{\gamma}_j) = u^n_0$,
   c) if $u^n_0 \in [0, \sigma]$ then $f^{-1}(\hat{\gamma}_j) \in [0, \sigma]$,
   with $j = 1, 2$.

Recall that u^n_m indicates a macroscopic variable and it represents a density.

10.2 Examples

This section deals with some examples of special situations.

10.2.1 Bottleneck

The simplest application is given by a bottleneck, which can be represented as a junction with one incoming road, one outgoing road and with the entering road having a maximum flux larger than the exiting ones. Therefore, we set the flux in the first part of the street to be equal to

$$f_1(\rho) = \rho(1-\rho), \quad \rho \in [0,1], \tag{10.2.24}$$

while, in the narrowest part of the street, the flux considered is

$$f_2(\rho) = \rho\left(1 - \frac{3}{2}\rho\right), \quad \rho \in [0, 2/3]. \tag{10.2.25}$$

The maximum for the fluxes is unique:

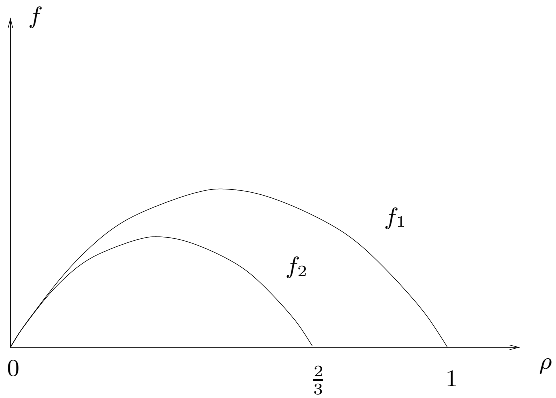


Fig. 10.1. The flux functions $f_1(\rho)$ and $f_2(\rho)$.

$$f_1(\sigma_1) = \max_{[0,1]} f_1(\rho) = \frac{1}{4}, \quad \text{with } \sigma_1 = \frac{1}{2}, \tag{10.2.26}$$

$$f_2(\sigma_2) = \max_{[0,2/3]} f_2(\rho) = \frac{1}{6}, \quad \text{with } \sigma_2 = \frac{1}{3}. \tag{10.2.27}$$

A key role is played by the separation point between the two parts of the road, say S. Indicate by ρ_s the density on the left respect of S (that belongs to the widest part of the street) and by ρ_d the density on the right of S.

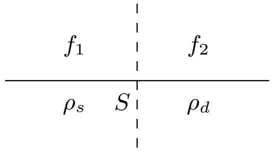


Fig. 10.2. Interface at the bottleneck.

The maximal fluxes f_1 and f_2 are computed following the rules, respectively

$$f_1^{max}(\rho) = \begin{cases} f_1(\rho_s) & \text{if } \rho_s \leqslant \sigma_1, \\ f_1(\sigma_1) & \text{if } \rho_s \geqslant \sigma_1, \end{cases}$$

$$f_2^{max}(u) = \begin{cases} f_2(\sigma_2) & \text{if } \rho_d \leqslant \sigma_2, \\ f_2(\rho_d) & \text{if } \rho_d \geqslant \sigma_2 \end{cases}$$

and the flux at the intersection point between the two intervals is obtained taking the minimum

$$\gamma = \min\{f_1^{max}(\rho_s), f_2^{max}(\rho_d)\}. \tag{10.2.28}$$

As the maximum density allowed in the second part is given by $\sigma_2 = \frac{1}{6}$, the creation of queues occurs when the density on the first road verifies

$$\rho(1-\rho) = \frac{1}{6} \Longleftrightarrow \bar{\rho} = \frac{1 - \sqrt{\frac{1}{3}}}{2} \simeq 0.21 \ . \tag{10.2.29}$$

Then, when $\rho_{1,b} < \bar{\rho}$ (recall that $\rho_{1,b}$ is the car density entering the largest road) there is no formation of shocks propagating backwards.

10.2.2 Traffic Circle

As in Chapter 8, consider a simple network representing a traffic circle composed by four roads, named I_1, I_2, I_3, I_4, the first two incoming in the circle and the other two outgoing. In addition there are four roads $I_{1R}, I_{2R}, I_{3R}, I_{4R}$ that form the circle as in Figure 10.3.

10.3 Tests

In this section we present some numerical tests performed with the schemes previously introduced: the Godunov's scheme (G), the three-velocities kinetic scheme of first order ($3VK_1$) and the three-velocities kinetic method ($3VK_2$) with $\lambda_3 = -\lambda_1 = 1.0$ and $\lambda_2 = 0$. In general the three-velocities models work better than the two-velocities ones. We introduce the formal numerical order γ of a numerical method as an average in the following way:

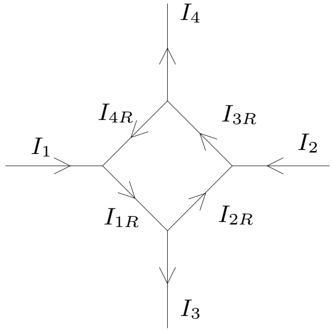


Fig. 10.3. Traffic circle.

$$\gamma = \frac{1}{R}\sum_{r=1}^{R}\gamma_r, \tag{10.3.30}$$

where

$$\gamma_r = \log_2\left(\frac{e^r(1)}{e^r(2)}\right), \quad r = 1,\ldots,R, \tag{10.3.31}$$

with r the index of roads composing the network. The L^1-error on each road is

$$e^r(p) = \frac{h}{p}\sum_{j=0,\ldots,pN}\left|w_j^{pM}\left(\frac{h}{p}\right) - w_{2j}^{pM}\left(\frac{h}{2p}\right)\right|, \quad p = 1,2,\ r = 1,\ldots,R, \tag{10.3.32}$$

where $w_m^M(h)$ denotes the numerical solution obtained with the space step discretization equal to h, calculated in x_m at the final time $t_M = T$. Note that if $e^r(p) = 0$, then we set $\gamma_r = 0$ and $R = R - 1$. The total L^1-error is

$$TOT_{err} = \sum_{r=1}^{R} e^r(1)\ . \tag{10.3.33}$$

The quantity R indicates the number of roads in the network and $w_m^M(h)$ denotes the numerical solution obtained with the space step discretization equal to h, computed in x_m at the final time $t_M = T$.

10.3.1 Bottleneck

We assume to deal with a road of length 2 parameterized by the interval $[0, 2]$ and that the separation point is placed in the middle of the road, namely at $x = 1$.

The next tables provide a comparison between the three methods in terms of L^1-error and order of convergence γ.

Test B1. We take the following initial and boundary data

$$\rho_1(0,x) = 0.66, \qquad \rho_2(0,x) = 0.66, \tag{10.3.34}$$

$$\rho_{1,b}(t,0) = 0.25\ .$$

Since the initial value 0.66 is very close to the maximum value that can be absorbed by road 2, after a short time, namely $T = 2$, the formation of a traffic jam can be observed, see Figure 10.4.

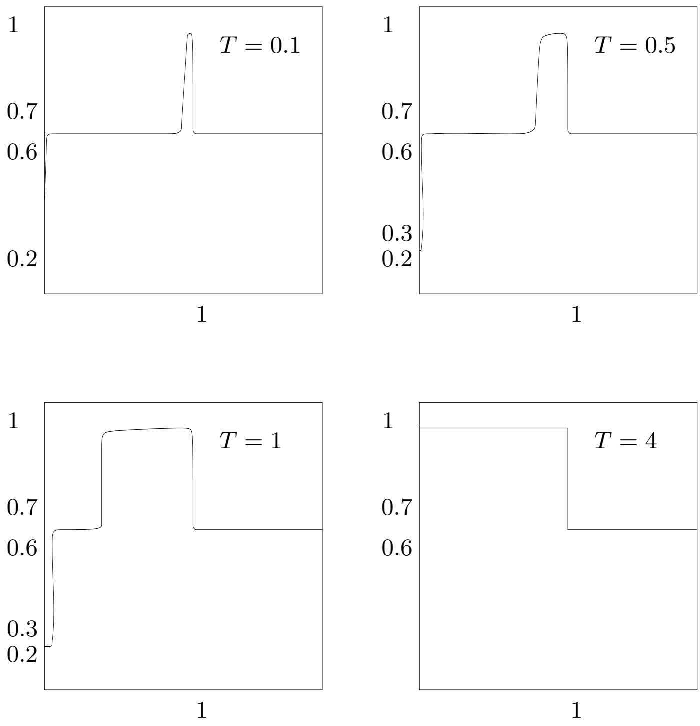


Fig. 10.4. Evolution in time for data (10.3.34) computed by $3VK_2$ scheme, $h = 0.0125$.

Test B2. Let us assume the road is initially empty and take the following initial and boundary data

$$\rho_1(0,x) = \rho_2(0,x) = 0, \tag{10.3.35}$$

$$\rho_{1,b}(t,0) = 0.4\ . \tag{10.3.36}$$

Since $\rho_{1,b} > \bar{\rho} \simeq 0.21$, also in this case there is a jam formation, , see Figure 10.5.

We report tables of orders and errors for both tests.

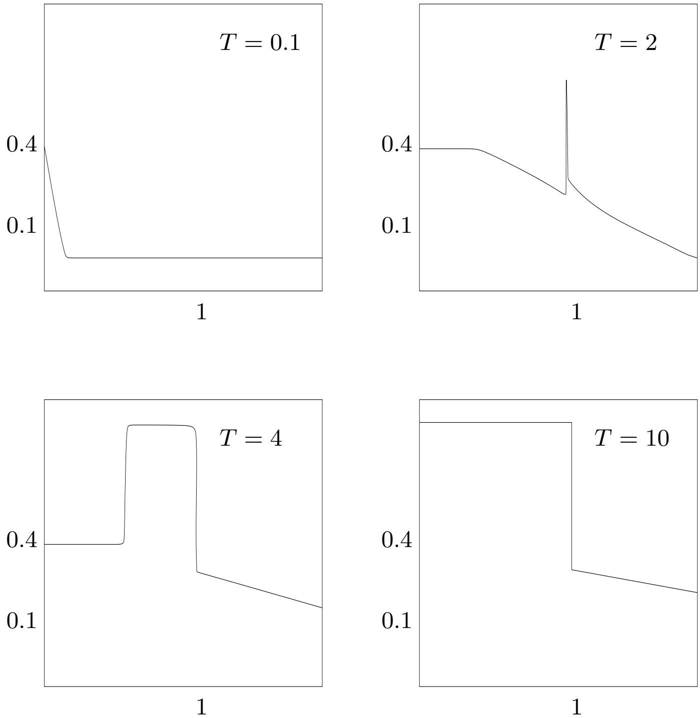


Fig. 10.5. Evolution in time for data (10.3.35) computed by $3VK_2$ scheme, $h = 0.0125$.

	G		$3VK_1$		$3VK_2$	
h	γ	L^1 Error	γ	L^1 Error	γ	L^1 Error
0.1	1.51554	3.347e-002	1.14981	2.886e-002	1.19519	2.931e-002
0.05	0.89752	1.170e-002	0.83645	1.301e-002	0.92098	1.280e-002
0.025	0.58367	6.285e-003	0.85088	7.284e-003	0.75549	6.761e-003
0.0125	1.22648	4.194e-003	1.16427	4.038e-003	1.29260	4.005e-003
0.00625	0.65763	1.792e-003	0.83753	1.802e-003	0.73386	1.635e-003
0.003125	1.50268	1.136e-003	1.12176	1.008e-003	1.50429	9.830e-004

TABLE B1-1: Orders and errors of the approximation schemes Godunov (G), Kinetic of first order ($3VK_1$) and of second order ($3VK_2$) for data (10.3.34), $T = 0.5$.

	G	$3VK_1$	$3VK_2$
h	L^1 Error	L^1 Error	L^1 Error
0.1	2.07651e-002	2.19038e-002	2.41712e-002
0.05	1.25376e-002	1.45365e-002	1.35243e-002
0.025	8.38778e-003	8.07708e-003	8.00970e-003
0.0125	3.58458e-003	3.60392e-003	3.26967e-003
0.00625	2.27234e-003	2.01675e-003	1.96603e-003
0.003125	8.01899e-004	9.26764e-004	8.49835e-004

TABLE B1-2: Errors of the approximation schemes Godunov (G), Kinetic of first order ($3VK_1$) and of second order ($3VK_2$) for data (10.3.34), $T = 1.0$.

	G		$3VK_1$		$3VK_2$	
h	γ	L^1 Error	γ	L^1 Error	γ	L^1 Error
0.1	0.65705	1.841e-002	0.65705	1.841e-002	0.62999	1.693e-002
0.05	0.67659	1.167e-002	0.67659	1.168e-002	1.56047	1.094e-002
0.025	0.70677	7.305e-003	0.70676	7.306e-003	0.50455	3.709e-003
0.0125	0.73821	4.476e-003	0.73821	4.476e-003	1.72696	2.614e-003
0.00625	0.76816	2.683e-003	0.76816	2.683e-003	0.81448	7.898e-004
0.003125	0.79447	1.575e-003	0.79447	1.575e-003	0.70529	4.491e-004

TABLE B2-1: Orders and errors of the approximation schemes Godunov (G), Kinetic of first order ($3VK_1$) and of second order ($3VK_2$) for data (10.3.35), $T = 1$.

	G	$3VK_1$	$3VK_2$
h	L^1 Error	L^1 Error	L^1 Error
0.1	2.16316e-002	2.18455e-002	1.69308e-002
0.05	7.10040e-003	1.09717e-002	1.09403e-002
0.025	4.70270e-003	5.44031e-003	3.70921e-003
0.0125	2.48223e-003	2.61377e-003	2.61455e-003
0.00625	1.09907e-003	8.57023e-004	7.89821e-004
0.003125	5.80967e-004	3.61744e-004	2.75442e-004

TABLE B2-2: Errors of the approximation schemes Godunov (G), Kinetic of first order ($3VK_1$) and of second order ($3VK_2$) for data (10.3.35), $T = 4$.

From the analysis of the previous Tables we can see that both $3VK_1$ and $3VK_2$ perform better than the Godunov scheme. In fact, the kinetic schemes show a good stability even after the interaction at the junction.

10.3.2 Traffic Circle

The numerical solutions have been generated by the ($3VK_2$) method for $h = 0.025$ and $CFL = 0.5$.

Fix the following initial and boundary data

$$\begin{aligned} &\rho_1(0,x) = 0.25,\ \rho_2(0,x) = 0.4,\ \rho_3(0,x) = \rho_4(0,x) = 0.5,\\ &\rho_{1R}(0,x) = 0.5,\ \rho_{2R}(0,x) = 0.5,\ \rho_{3R}(0,x) = \rho_{4R}(0,x) = 0.5,\\ &\rho_{1,b}(t,0) = 0.25, \quad \rho_{2,b}(t,0) = 0.4. \end{aligned} \tag{10.3.37}$$

The distribution coefficients, namely ($\alpha_{1R,3}, \alpha_{1R,2R}, \alpha_{3R,4}, \alpha_{3R,4R}$), are assumed to be constant and are all equal to $\alpha = 0.5$.

First, set the right of way parameters $q_1 = q(1, 4R, 1R) = 0.25$, $q_2 = q(2, 2R, 3R) = 0.25$. The fixed values imply that road I_{4R} is the through street respect to road I_1 and road I_{2R} is the through street respect to I_1. The

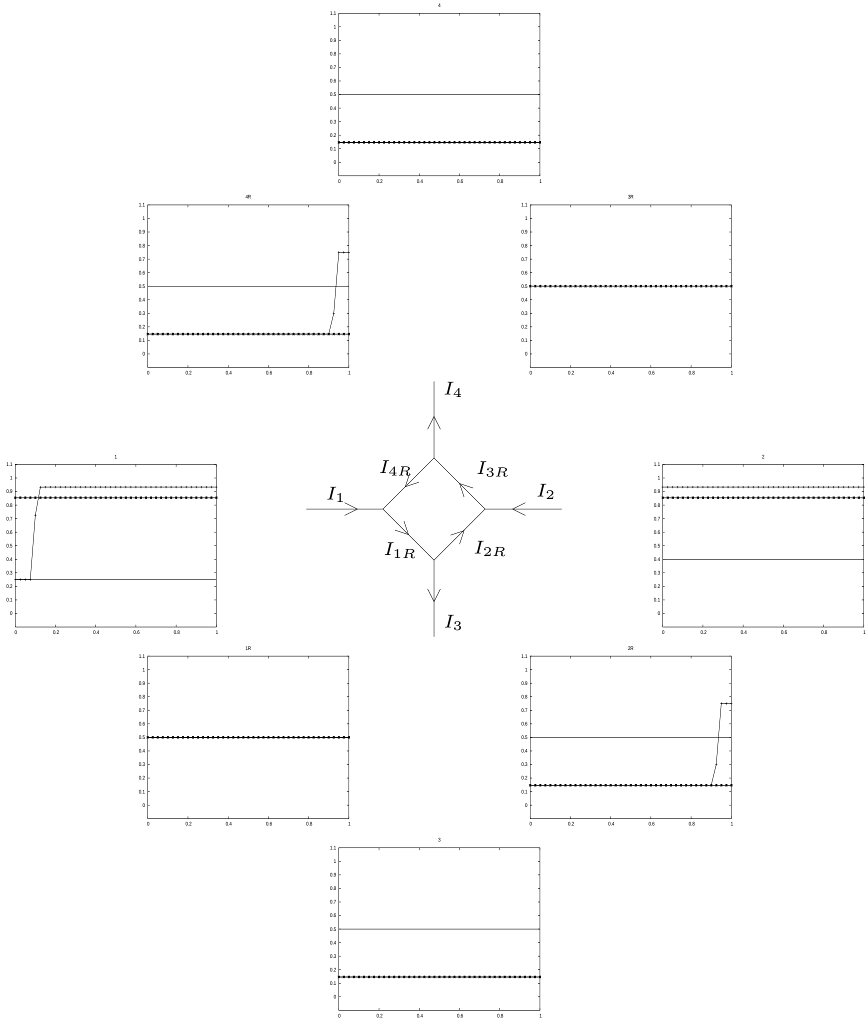


Fig. 10.6. Traffic circle with $q_1 = q_2 = 0.25$.

evolution in time of traffic is reported in Figure 10.6 (for the legend see the end of the Section). Observe that at time $t = 5$ shocks are generated on the entering roads I_1 and I_2, while rarefaction waves in the direction of traffic are created on roads I_{4R}, I_{2R}, I_3, I_4. Roads I_{1R} and I_{3R} maintain the same level of density. At $t = 10$ rarefaction waves traveling in the sense of traffic produce a decrease in the car density on roads I_{4R}, I_{3R}, I_3, I_4. On entering roads I_1 and I_2, the effect of shocks traveling backwards is a considerable increase of the density and again, roads I_{1R} and I_{3R} maintain the same configuration, which corresponds to the maximum flux. At time $T = 40$ the roads entering in the circle have an high value of density as they wait at the junctions, while densities of roads in the circle are lowered due to the fact that traffic is flowing towards the outgoing roads I_3 and I_4. We can observe that starting from the same configuration (10.3.37) but setting differently the right of way parameters, traffic within the circle is fluid and is distributed between the outgoing roads.

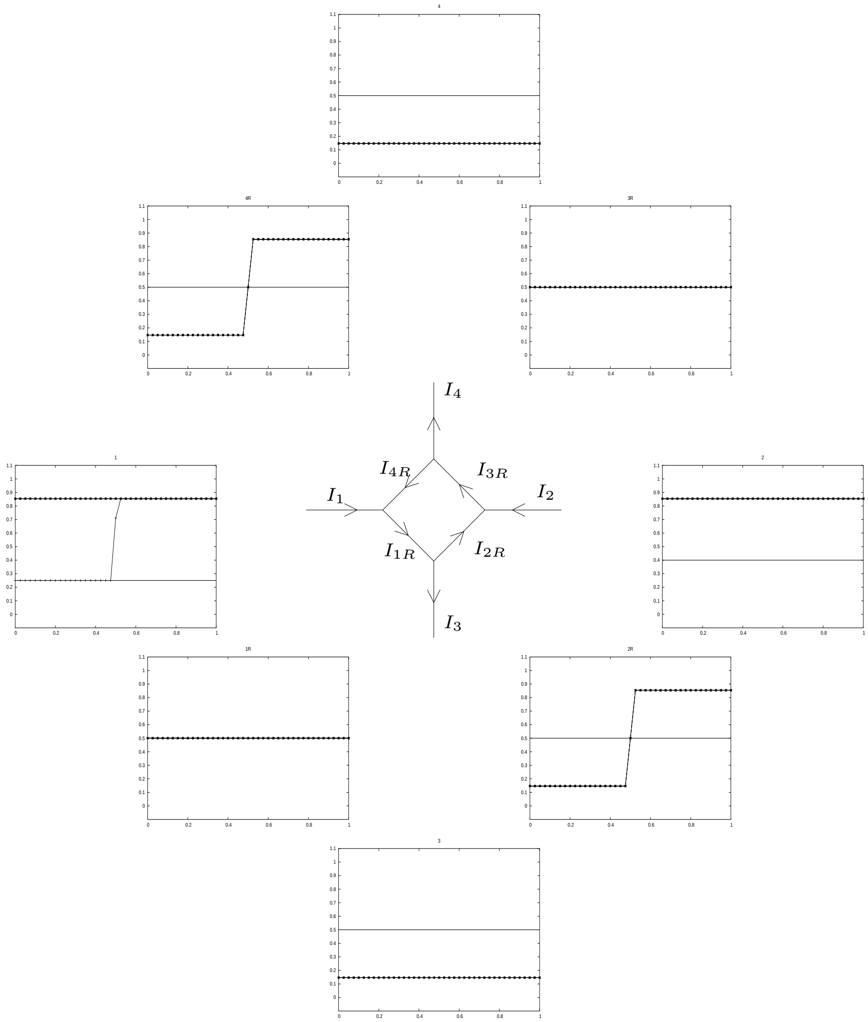


Fig. 10.7. Traffic circle with $q_1 = q_2 = 0.5$.

Figure 10.7, obtained for data (10.3.37) and $q_1 = q_2 = 0.5$, shows a situation quite similar to that in Figure 10.6. The difference is represented by the values of density on the roads I_{2R} and I_{4R} that reveal a shock formation with zero speed. As a consequence, the percurrence time of the circle from road I_1 to road I_4 is higher than in the case depicted in Figure 10.6. In particular, let δ be the portion of road I_{2R} at the lowest value of density, i.e. 0.15, and $1-\delta$ the other portion of the same road, we can estimate the the percurrence time from road I_1 to road I_4. In the first case is

$$\frac{1}{0.5}+\frac{1}{0.85}+\frac{1}{0.5}\sim 5.17$$

while here (with $\delta = 0.5$) we get

$$\frac{1}{0.5}+\frac{\delta}{0.85}+\frac{1-\delta}{0.15}+\frac{1}{0.5}\sim 7.92$$

and the difference between the previous and the current case is

$$\Delta t=\frac{1-\delta}{0.15}-\frac{1-\delta}{0.85}=(1-\delta)\frac{80}{17}.$$

If the right of way parameters are

$$q_1 = q(1,4R,1R) = 0.75, \quad q_2 = q(2,2R,3R) = 0.75,$$

then road I_1 is the through street respect to road I_{4R} and road I_2 is the through street respect to I_{2R}. As before, the distribution coefficients are assumed to be constant and all equal to $\alpha = 0.5$. The evolution in time of traffic densities is described in Figure 10.8. One can observe that at time $t = 1.5$ the chosen right of way parameters provoke shocks propagating backwards along roads I_{2R} and I_{4R} and consequently a shock is created on road I_2. Successively, the density on roads I_{4R}, I_{2R} increases and shocks are propagating backwards on roads I_{1R} and I_{3R}. Roads I_3 and I_4 show a very low density of cars. At $T = 40$ densities on the incoming roads and within the circle (all equal to the maximum value $\rho_{\max} = 1$), represent a situation of traffic jam, the so called bumper-to-bumper traffic. This means that no cars can exit the circle, as showed by the fact that roads I_3 and I_4 are empty. Hence, in that case, the choice of the right of way parameter determines a situation of completely blocked traffic.

Figures 10.6, 10.7 and 10.8, show the evolution in time of the density for the different choices of the right of way parameter with the following legend:

In the same framework, we can also analyze portions of urban network. For some simulations, see [24].

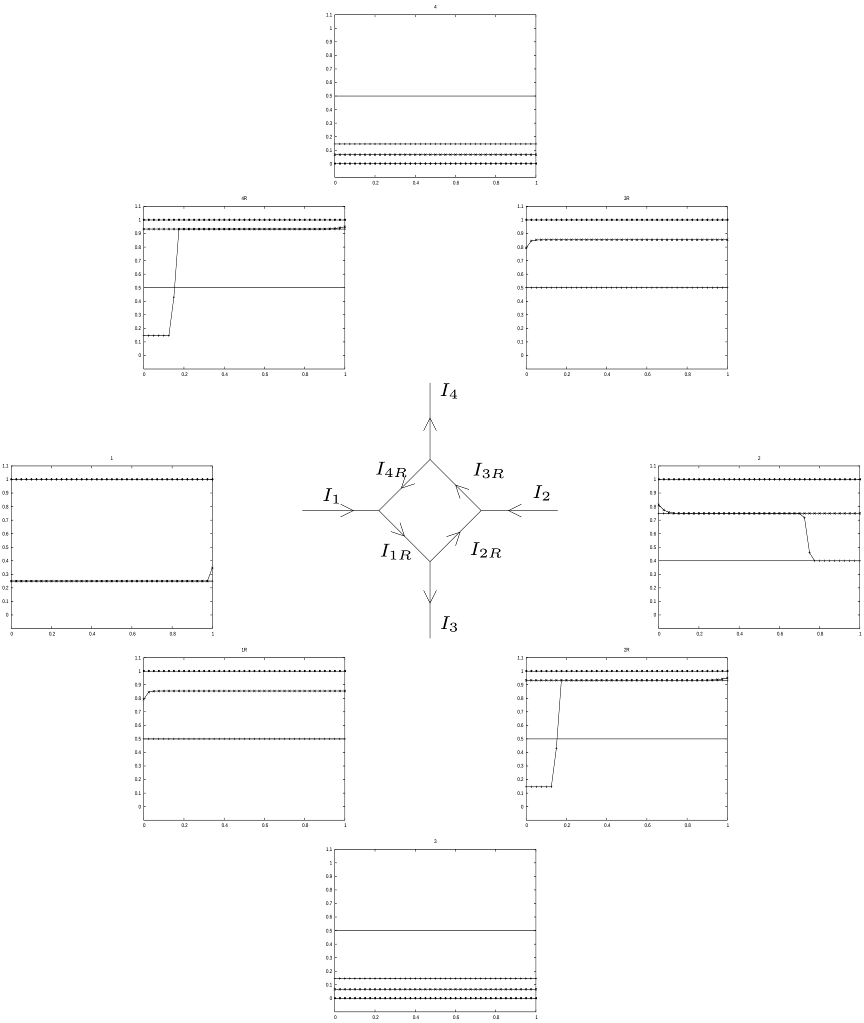
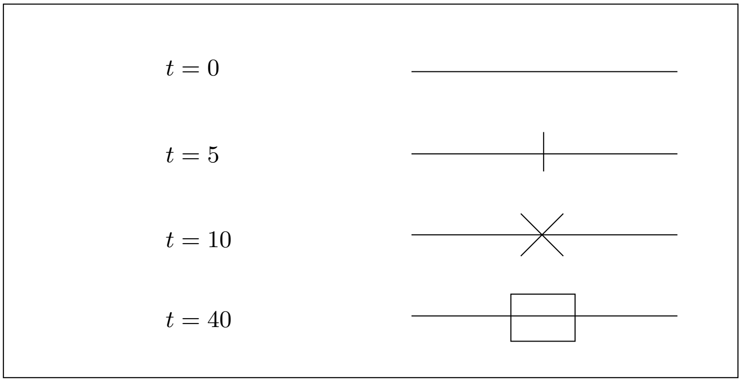


Fig. 10.8. Traffic circle with $q_1 = q_2 = 0.75$.


$t = 0$
$t = 5$
$t = 10$
$t = 40$

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